A study of the electro- and magnetostatic equations of the ECE Theory

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Important note

This paper is not yet completely finished,
i.e the sections 2.5.2 , 3.5.1 , 3.6 and 3.7 are still empty

Abstract / Summary

The ECE Theory (www.aias.us) represents an unified field theory of electromagnetism, gravitation, weak and strong interaction. It leads, among others, to a set of electromagnetic equations which represent an extension of the Maxwell or Maxwell-Heaviside equations of textbook electrodynamics. This extension is based on the existence of certain physical quantities which are not present in textbook electrodynamics. They are called vector and scalar spin connection and are related to the spinning or torsion of space-time. Several ECE papers report on solutions of the electromagnetic ECE equations which show resonance-like peaks in the potential. According to the ECE Theory these resonances can be used to extract usable energy from space-time.

The purpose of this paper is to clarify the commonalities and differences between ECE and textbook electromagnetism. For the sake of simplicity this work deals only with electro- and magnetostatics whereby three different sets of equations were investigated, namely the mere electrostatic ECE equations, the former electro- and magnetostatic ECE equations and the latest electro- and magnetostatic ECE equations. The latest equations represent a modification of the former equations. That modification emerged from recent results of the ECE Theory which report on the discovery of additional equations, the so-called antisymmetry contraints.

The conclusions of the study of the mere electrostatic ECE equations are the following. Electrostatics alone is not sufficient to generate resonances in the electric potential or field, i.e. novel detectable features can be expected only for more complex scenarios such as electrodynamics. However, if we assume that the so-called homogeneous current $\vec{j}\,(\vec{r})$ does not vanish, then even in electrostatics the potential or field may show novel features which are not possible in textbook electrostatics. This statement results from a general solution of the electrostatic ECE equations for

$\vec{j}(\vec{r}) \neq 0$ which is presented in this paper. Although it is not clear how to generate a homogeneous current, which is according to the ECE Theory related to gravitation, its influence on an electric potential or field represents an interesting effect, at least theoretically.

The results of the study of the electro- and magnetostatic ECE equations can be summarized as follows. The relevant quantities which appear in the electro- and magnetostatic ECE equations are the charge density $\rho(\vec{r})$, current density $\vec{J}(\vec{r})$, scalar potential $\phi(\vec{r})$, vector potential $\vec{A}(\vec{r})$, scalar spin connection $\omega_0(\vec{r})$ and vector spin connection $\vec{\omega}(\vec{r})$. By the introduction of another scalar potential $g(\vec{r})$ and another vector potential $\vec{V}(\vec{r})$ the original electro- and magnetostatic ECE equations can be transformed into a set of equations which do not contain the spin connection any more. The transformed equations allow a direct comparison between ECE and textbook electro- and magnetostatics. The transformed equations decompose into two sets of equations and physical quantities which indicate the existence of two different but interconnected physical levels or realities. In this paper we call them level I and level II. Level I corresponds to the physics of textbook electro- and magnetostatics. Level II can be considered as an underlying physical reality which is more subtle than that of textbook electro- and magnetostatics. The relevant physical quantities and their relations among each other are

- scalar potential $\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r})$
- electric field $\vec{E}(\vec{r}) = -\vec{\nabla}\beta(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r})$
- vector potential $\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r})$
- magnetic field $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{\Lambda}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r})$

whereby the potential or field on the left is a level I quantity, whereas the both potentials or fields on the right are level II quantities. With respect to the level I quantities $\beta(\vec{r})$, $\vec{E}(\vec{r})$, $\vec{\Lambda}(\vec{r})$ and $\vec{B}(\vec{r})$ textbook and ECE electro- and magnetostatics are equivalent. In ECE electro- and magnetostatics, however, these level I potentials and fields emerge from a difference of two level II quantitites which both depend on $\vec{r}$.

It should be emphasized that ECE and textbook electromagnetism usually use the same symbol for the scalar and vector potential, namely ϕ and and $\vec{A}$, respectively. However, as revealed by the transformed equations, they do not represent the same physical quantity and have to be distinguished. In this paper ϕ and $\vec{A}$ denotes the scalar and vector potential which appears in the original ECE equations, whereas β and $\vec{\Lambda}$ corresponds to the scalar and vector potential of textbook electro- and magnetostatics, respectively.

The transformed equations reveal distinct differences between the equations and quantities of level I and II. The level I potentials $\beta(\vec{r})$ and $\vec{\Lambda}(\vec{r})$ are specified by four decoupled, linear, second-order differential equations. Thus the level I electric and magnetic quantities or phenomena are decoupled from each other. The level I equations and quantities are well-known and represent the physics of textbook electro- and magnetostatics.

Concerning the former set of the electro- and magnetostatic ECE equations, the level II potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ are specified by four coupled, non-linear, first-order differential equations. Thus on level II the electric and magnetic quantities or phenomena are coupled with each other.

Concerning the latest set of the electro- and magnetostatic ECE equations, the level II vector potential $\vec{A}(\vec{r})$ is specified by four coupled, non-linear, first-order differential equations which comprise exclusively magnetic quantities, whereas the level II scalar potential $\phi(\vec{r})$ is specified by electric quantities and $\vec{A}(\vec{r})$. Thus on level II the electric and magnetic quantities or phenomena are partly coupled with each other. The number of equations which specify $\vec{A}(\vec{r})$ and $\phi(\vec{r})$ are greater than four. Thus $\vec{A}(\vec{r})$ and $\phi(\vec{r})$ are possibly over-determined. Further studies are necessary to clarify if for any charge density $\rho(\vec{r})$ and any current density $\vec{J}(\vec{r})$ a solution exists.

For the former and latest set of the electro- and magnetostatic ECE equations, which specify the level II potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$, we present solutions for two special cases, namely for $\vec{E} = \vec{B} = 0$ as well as for $\vec{B} = 0$ and $\vec{E} \neq 0$ with any charge density $\rho(\vec{r})$. For the first case, i.e. $\vec{E} = \vec{B} = 0$, the solutions represent possible level II vacuum potentials and fields in the absence of level I fields. The number of possible solutions is infinite but they are not arbitrary. For the latest set of equations the presented vacuum vector potentials are curl-free which implies $\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = 0$. Therefore we raise for the latest set of equations the question if there are also vacuum solutions with $\vec{\nabla} \times \vec{A} \neq 0$ which would imply $\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) \neq 0$. For the second case, i.e. $\vec{B} = 0$ and $\vec{E} \neq 0$ with any charge density $\rho(\vec{r})$, we found for the former and the latest set of equations a common type of possible solutions for the level II scalar potential $\phi(\vec{r})$, namely

$$\phi(\vec{r}) = \left[a + b \ln\left(\frac{\beta(\vec{r})}{\beta_0} \right) \right] \beta(\vec{r}) \quad \text{whereby} \quad \beta(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|} \, d^3R$$

is the level I scalar potential of textbook electro- and magnetostatics and a , b and β_0 are constants.

The former set of the electro- and magnetostatic ECE equations appears as a consistent extension of textbook electro- and magnetostatics. The latest set of the electro- and magnetostatic ECE equations comprises the relation $\vec{E}(\vec{r}) = -\,2\,\vec{A}(\vec{r})\,\omega_0(\vec{r})$ which involves a potential inconsistency. This relation represents one of several ways to express the electric field in terms of other quantities. We consider the special case $\vec{B} = 0$ and it seems that this relation can only be satisfied if the electric field corresponds to that of a single point charge. It seems that for any other charge densitiy such as two point charges an appropriate scalar spin connection $\omega_0(\vec{r})$ does not exist. We briefly suggest some items whose consideration might lead to an elimination of this inconsistency, e.g. the assumption that $\omega_0(\vec{r})$ does not represent a scalar but a more complex quantity like a 3×3 matrix.

However, further studies, especially at a higher level of the ECE Theory which comprises field theory and differential geometry, are necessary to clarify this issue.

The transformed equations lead to the following significant questions: Are the level II potentials and fields, or their effects, physically detectable ? Do the level I and II quantities carry the same or different physical characteristics ? Which of three related quantities such as $\beta(\vec{r})$, $\phi(\vec{r})$ and $g(\vec{r})$ is the relevant quantity (in a specific context or problem) ? Concerning the latter question it seems that there are essentially two answers or views:

• The relevant quantity is the level I potential or field such as $\beta(\vec{r})$, whereas the two associated level II quantities such as $\phi(\vec{r})$ and $g(\vec{r})$ belong to a more subtle physical reality which is experimentally not yet explored.

• Another view is the following: The relevant quantity is one of the two level II potentials or fields such as $\phi(\vec{r}) = \beta(\vec{r}) + g(\vec{r})$ which represents a modification of the quantity of textbook electro- and magnetostatics. This view predicts a potential and field which is always different from that of textbook electro- and magnetostatics. This might appear unsatisfying, however it is maybe conceivable that the additional potential or field such as $g(\vec{r})$ develops its efficacy only under special conditions.

By means of a gedanken experiment and the feature that the level I potentials and fields emerge from a difference of two level II quantities we derive a tentative expression for the vacuum energy density in the absence of level I fields.

Also the electrodynamic ECE equations are briefly presented and discussed. We raise the question if the approach presented in this work can also be applied to the time-dependent ECE equations. In other words: Do also the time-dependent potentials and fields emerge from a difference of two potentials or fields which both depend on $\vec{r}$ and t ?

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1 The electrostatic ECE equations

1.1 The original equations and results and discussion of the transformed equations

Several papers about the ECE electrostatics report on the presence of resonances in the electric potential ϕ, see e.g. Refs. [1, 2, 3, 4, 5, 6]. The resonances in ϕ, which may result in an extraction of usable energy from space-time, are related to certain quantities which are not present in textbook electrostatics and electrodynamics. They are called spin connection and are related to a spinning space-time.

The electrostatic ECE equations in vector notation are given by [1] [2]

$$\Delta\phi \;-\; \vec{\nabla}\cdot\left(\vec{\omega}\,\phi\right) \;=\; -\frac{\rho}{\epsilon_0} \tag{1}$$

$$\vec{\nabla}\times\left(\vec{\omega}\,\phi\right) \;=\; 0 \tag{2}$$

$$\vec{E} \;=\; -\,\vec{\nabla}\phi \;+\; \vec{\omega}\,\phi \tag{3}$$

whereby $\Delta = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplace operator, $\rho = \rho(\vec{r})$ the charge density, $\phi = \phi(\vec{r})$ the scalar electric potential, $\vec{\omega} = \vec{\omega}(\vec{r})$ the vector spin connection, and $\vec{E} = \vec{E}\,(\vec{r})$ the electric field.

For $\vec{\omega} = 0$ Eqs. (1) $-$ (3) merge into the equations of textbook electrostatics and Eq. (1) yields the so-called Poisson equation

$$\Delta\phi = -\frac{\rho}{\epsilon_0} \tag{4}$$

For a spatially limited charge density, i.e. $\rho(\vec{r}) \to 0$ for $r \to \infty$, the well-known

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[1] Eqs. (1) $-$ (3) emerge from the former set of the electro- and magnetostatic ECE equations, i.e. Eqs. (77) $-$ (82), by $\vec{A} = 0$. Furthermore, Eqs. (1) $-$ (3) refer to the assumption that the so-called polarization index can be omitted, i.e. one polarization only, see e.g. Ref. [6].

[2] In this paper we use the same defintion of the sign of $\vec{\omega}$ as in Ref. [6] which is the most straightforward definition resulting from the ECE field equations [7]. In the context of resonances in ϕ many ECE papers, see e.g. Ref. [8], use another definition of the sign, i.e. $\vec{\omega}$ is replaced by $-\,\vec{\omega}$. That means Eqs. (1) $-$ (3) are replaced by $\Delta\phi + \vec{\nabla}\cdot\left(\vec{\omega}\,\phi\right) = -\frac{\rho}{\epsilon_0}$, $\vec{\nabla}\times\left(-\,\vec{\omega}\,\phi\right) = 0$ and $\vec{E} = -\,\vec{\nabla}\phi - \vec{\omega}\,\phi$. We note that the minus sign in the second equation must not be omitted when referring to Eq. (9) because $\vec{\omega}\,\phi = \vec{\nabla}\sigma$ has to be replaced by $-\,\vec{\omega}\,\phi = \vec{\nabla}\sigma$. We raise the question if this issue has been taken into account in the ECE papers. The ECE papers are focussed on the spin connection and do not use the scalar field $\sigma(\vec{r})$. However, it seems that a sign reversal of $\vec{\omega}$ in Eq. (2) has to be considered in some way in the process of the ECE calculations, even if Eq. (2) suggests to omit it because its right-hand side is equal to zero.

solution of Eq. (4) is given by [3] [4]

$$\phi(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{\mid \vec{r} - \vec{R} \mid}\, d^3R \tag{5}$$

For a point charge q located at a position $\vec{a}$, i.e. $\rho(\vec{R}) = q\,\delta(\vec{R} - \vec{a})$, whereby $\delta(\vec{R} - \vec{a}) = \delta(X - a_1)\,\delta(Y - a_2)\,\delta(Z - a_3)$ represents the Dirac delta function, Eq. (5) reproduces the well-known Coulomb law [5]

$$\phi(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{q}{\mid \vec{r} - \vec{a} \mid} \tag{6}$$

Eq. (5) represents the summation of the Coulomb potentials

$$d\phi(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{\rho(\vec{R})}{\mid \vec{r} - \vec{R} \mid}\, d^3R \tag{7}$$

of all charges $\rho(\vec{R})\, d^3R$ which are distributed in a spatially limited range.

For a given charge density $\rho(\vec{r})$ the corresponding electrostatic potential $\phi(\vec{r})$ and the spin connection $\vec{\omega}(\vec{r})$ results from the simultaneous solution of Eqs. (1) and (2). Now we consider their solution in the following way. Because of

$$\vec{\nabla} \times \left(\vec{\nabla}\sigma\right) = 0 \tag{8}$$

Eq. (2) is satisfied if

$$\vec{\omega}\,\phi = \vec{\nabla}\sigma \tag{9}$$

whereby $\sigma = \sigma(\vec{r})$ is a scalar field. Already at this point it is obvious that the product $\vec{\omega}\,\phi$ is not unambiguously defined because also $\vec{\omega}\,\phi = \vec{\nabla}\,(\sigma + \sigma\,')$ satisfies Eq. (2) whereby $\sigma\,' = \sigma\,'(\vec{r})$ is another scalar field. It should be noted that Eqs. (1)

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[3] We recall that $\Delta\,\frac{1}{\mid \vec{r} - \vec{R} \mid} = -\,4\,\pi\,\delta(\vec{r} - \vec{R})$ whereby $\delta(\vec{r} - \vec{R}) = \delta(x - X)\,\delta(y - Y)\,\delta(z - Z)$ represents the Dirac delta function.

[4] The most general solution of Eq. (4) is given by
$\phi(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{\mid \vec{r} - \vec{R} \mid}\, d^3R \,+\, \phi_{hom}(\vec{r})$
whereby $\phi_{hom}(\vec{r})$ is a solution of the Laplace equation $\Delta\phi_{hom} = 0$

[5] We note that the Coulomb law is also related to the Gauss law which is just another way to express the Coulomb law. The Gauss law states that the total flux through a closed surface (which is given by a surface integral) is $\frac{Q}{\epsilon_0}$ whereby Q is the total charge located within the closed surface. Thus, if there are only charges which are exclusively located outside of a closed surface, then the total flux through this closed surface is zero.

and (2) represent four equations to determine the four quantities ϕ and $\vec{\omega}$. Nevertheless, due to the intrinsic ambiguousness of rotational fields or operators, the solutions of these equations comprise a (nearly) arbitrary term.

In other ECE papers the spin connection $\vec{\omega} = \vec{\omega}(\vec{r})$ represents an adjustable function and the scalar field $\sigma = \sigma(\vec{r})$ is not considered. In this paper we have introduced the scalar field $\sigma = \sigma(\vec{r})$ which is apriori an arbitrary function whose associated spin connection can be calculated. By inserting Eq. (9) in Eq. (1) we obtain

$$\Delta\phi - \vec{\nabla} \cdot \left(\vec{\nabla}\sigma\right) = -\frac{\rho}{\epsilon_0} \tag{10}$$

and thus

$$\Delta\phi = -\frac{\rho}{\epsilon_0} + \Delta\sigma \tag{11}$$

or

$$\Delta(\phi - \sigma) - \frac{\rho}{\epsilon_0} \tag{12}$$

or

$$\Delta\beta = -\frac{\rho}{\epsilon_0} \tag{13}$$

whereby

$$\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r}) \tag{14}$$

Thus, at least formally, the arbitrary function $\sigma(\vec{r})$ and the quantity $\beta(\vec{r})$ represent scalar potentials, whereas the quantity $\epsilon_0\,\Delta\sigma$ has the meaning of a charge density. Eq. (13) is equivalent to Eq. (4) of textbook electrostatics.

According to Eqs. (4) and (5) the solution of Eqs. (11) – (13) for a spatially limited charge density $\rho(\vec{r})$ can be represented by [6]

$$\phi(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R}) - \epsilon_0 \Delta\sigma(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \tag{15}$$

$$\phi(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \, + \, \sigma(\vec{r}) \tag{16}$$

$$\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \tag{17}$$

The integral term in Eqs. (16) and (17), and thus also the potential $\beta(\vec{r})$, is equal to the potential of textbook electrostatics. Eqs. (16) and (17) suggest two different views how to consider the description of electrostatics by ECE Theory, namely

[6] For the sake of simplicity we have omitted in Eqs. (15) – (17) on the right-hand side the solution of the corresponding Laplace equation, see footnote 4 on page 7.

1. Eq. (16) suggests that $\phi(\vec{r})$ corresponds or is related to the potential of textbook electrostatics. However, it depends also on the arbitrary potential $\sigma(\vec{r})$, i.e. its specific form is not determined by the electrostatic ECE equations (1) and (2).

2. Eqs. (13) and (17) shows that the potential $\beta(\vec{r})$ is equal to that of textbook electrostatics. Thus, on the level of the potential $\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r})$ textbook and ECE electrostatics are equivalent. In ECE electrostatics, however, the potential $\beta(\vec{r})$ arises from a difference between two potentials which both depend on $\vec{r}$, whereas in textbook electrostatics the potential results from one spatially dependent function.

We will discuss these both views somewhat later in more detail, however alreday at this place it seems that the second view represents the more plausible interpretation.

We emphasize that in ECE and textbook electrostatics usually the same symbol ϕ for the potential is used. However, especially from the perspective of Eq. (17), they do not represent the same physical quantitiy and have to be distinguished. In this paper ϕ represents the potential which appears in the original electrostatic ECE equations (1) $-$ (3) and β corresponds to the potential used in textbook electrostatics. According to Eqs. (16) and (17) the case of $\sigma(\vec{r}) = constant$ corresponds to the scenario of textbook electrostatics.

By inserting the potential (16) into Eq. (9) as well as Eqs. (16) and (9) into Eq. (3) we get its associated spin connection $\vec{\omega}$ and electric field $\vec{E}$, namely

$$\vec{\omega}\,(\vec{r}) = \frac{\vec{\nabla}\sigma(\vec{r})}{\displaystyle\iiint \frac{\rho(\vec{R})}{4\,\pi\,\epsilon_0\,|\,\vec{r}\,-\,\vec{R}\,|}\;d^3R \;+\; \sigma(\vec{r})} \tag{18}$$

$$\begin{aligned} \vec{E}\,(\vec{r}) &= \vec{E}_\phi\,(\vec{r}) - \vec{E}_\sigma\,(\vec{r}) = -\,\vec{\nabla}\left[\phi(\vec{r}) - \sigma(\vec{r})\right] = -\,\vec{\nabla}\beta(\vec{r}) \\ &= \frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{(\vec{r}\,-\,\vec{R})\,\rho(\vec{R})}{|\,\vec{r}\,-\,\vec{R}\,|^{\,3}}\;d^3R \end{aligned} \tag{19}$$

whereby we have introduced the electric fields $\vec{E}_\phi$ and $\vec{E}_\sigma$ by

$$\vec{E}_\phi = -\,\vec{\nabla}\phi \tag{20}$$

$$\vec{E}_\sigma = -\,\vec{\nabla}\sigma \tag{21}$$

In contrast to the integral representation of the potential $\phi(\vec{r})$, see Eq. (16), the integral representation of the electric field $\vec{E}\,(\vec{r})$ does not depend on $\sigma(\vec{r})$ and is equal to that of textbook electrostatics. Thus, on the level of the electric field $\vec{E}\,(\vec{r})$

the ECE and textbook electrostatics are equivalent. However, in contrast to the latter, the electric field $\vec{E}(\vec{r})$ in ECE electrostatics arises from a difference between two electric fields which both depend on $\vec{r}$ and from a difference between two potentials which both depend on $\vec{r}$.

Eq. (18) indicates that the spin connection $\vec{\omega}(\vec{r})$ represents a nearly arbitrary quantity because it is a function of the arbitrary potential $\sigma(\vec{r})$.

According to Eqs. (14) $-$ (21) the case of

$$\sigma(\vec{r}) = c \tag{22}$$

, whereby c is a constant, implies

$$\beta(\vec{r}) = \phi(\vec{r}) - c \tag{23}$$

$$\vec{\omega}(\vec{r}) = 0 \tag{24}$$

$$\vec{E}_\sigma(\vec{r}) = 0 \tag{25}$$

$$\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) = - \vec{\nabla}\phi(\vec{r}) = - \vec{\nabla}\beta(\vec{r}) \tag{26}$$

and corresponds to the scenario of textbook electro- and magnetostatics.

As already mentioned above, Eqs. (16) and (17) suggest two different perspectives how to view the description of electrostatics by ECE Theory. Let's first consider that which is suggested by Eq. (16). Although $\sigma(\vec{r})$ represents an apriori arbitrary function, some restrictions for $\sigma(\vec{r})$, $\phi(\vec{r})$ and $\vec{\omega}(\vec{r})$ result from the requirements for a spatially limited charge density $\rho(\vec{r})$, namely

$$\sigma(\vec{r}) \rightarrow 0 \quad \text{for} \quad r \rightarrow \infty \tag{27}$$

$$\phi(\vec{r}) \rightarrow 0 \quad \text{for} \quad r \rightarrow \infty \tag{28}$$

$$\vec{\omega}(\vec{r}) \rightarrow 0 \quad \text{for} \quad r \rightarrow \infty \tag{29}$$

In spite of these restrictions the number of choices for the function $\sigma(\vec{r})$ is still infinite. This appears to be an unsatisfactory situation because, according to Eq. (16), the nearly arbitrary function $\sigma(\vec{r})$ enters in the electric potential $\phi(\vec{r})$. Several ECE papers, see e.g. Refs. [1, 2, 3, 4, 5], consider such functions $\vec{\omega}(\vec{r})$ for which ϕ displays resonance features with respect to $\vec{r}$ and/or parameters associated with ρ such as $\vec{k}$ for

$$\rho = \rho_0 \, \cos(\vec{k} \cdot \vec{r}) \tag{30}$$

However, the ECE electrostatics in terms of Eqs. (1) and (2) implies the presence of a nearly arbitrary function $\sigma(\vec{r})$ in the potential $\phi(\vec{r})$. Thus it does not provide the physical or experimental conditions which favor the occurrence of resonance features in the potential ϕ. Therefore we conclude that the ECE electrostatics in terms of Eqs. (1) and (2) is not sufficient to create resonance phenomena in the potential ϕ. We note that resonance phenomena are usually associated with dynamic and time-dependent phenomena [7] [8] [9]. Therefore the time-dependent electrodynamic ECE equations appear as more appropriate candidates for resonance phenomena. The presence of resonances in those equations are already reported, see e.g. Ref. [12].

The interpretation of ECE electrostatics in terms of Eq. (16) appears unsatisfactory because of the presence of a nearly arbitrary function $\sigma(\vec{r})$ in the electric potential $\phi(\vec{r})$. However, as already mentioned, there is another and more plausible way how to view the description of electrostatics by ECE Theory, namely that which is suggested by Eqs. (17) and (19).

According to Eqs. (17) and (19) textbook and ECE electrostatics are equivalent on the level of the potential $\beta(\vec{r})$ and field $\vec{E}(\vec{r})$. This means that $\beta(\vec{r})$ and $\vec{E}(\vec{r})$ is identical with the potential and field of textbook electrostatics, respectively. In ECE electrostatics, however, the potential $\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r})$ and field $\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_\sigma(\vec{r})$ emerge from a difference between two functions which both depend on $\vec{r}$, whereas in textbook electrostatics they appear as one spatially dependent function which corresponds to the case of $\sigma(\vec{r}) = constant$. Thus the potential $\phi(\vec{r})$ which appears in the electrostatic ECE equations (1) $-$ (3) is not identical with the potential $\beta(\vec{r})$ of textbook electrostatics. The single potentials $\phi(\vec{r})$ and $\sigma(\vec{r})$ and single fields $\vec{E}_\phi(\vec{r})$ and $\vec{E}_\sigma(\vec{r})$, which are not specified by the electrostatic ECE equations (1) $-$ (3), point to a possible existence of a physical reality beyond that of textbook electrostatics. We raise the questions if these single potentials and fields are physically measurable and if they cause other physical effects than those of $\beta(\vec{r})$ and $\vec{E}(\vec{r})$. Possibly, the single potentials and fields, or their effects, are physically not detectable (with present technology) or their experimental verification requires special circumstances.

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[7] It should be mentioned that on the microscopic level nothing is static, everything is in permanent motion. Therefore the static case represents an approximation which may be sufficient for some macroscopic systems or considerations.

[8] In Global Scaling $-$ a new and holistic approach in science [9, 10] $-$ resonance features may appear on every physical scale, even when no macroscopic time dependence is involved. However, a discussion of this issue is beyond the scope of this paper

[9] Concerning the question if usable energy can be extracted from the space-time or quantum vacuum in electro- and/or magnetostatic situations, we refer to the interesting work of C. W. Turtur, see Ref. [11]. It reports on the conversion of vacuum energy into mechanical energy. A rotational movement of a rotor was achieved by the presence of an electrostatic field which was created by a high voltage (these experimental conditions might be considered as electrostatic or quasi-electrostatic). It was clearly shown that the mechanical power P_{mech} of the rotational movement is much higher than the electric power P_{el} to maintain the electric field (in practice P_{el} is not zero because at high voltages there are always some small leakage losses). Thus the difference $P_{mech} - P_{el} > 0$ is extracted from the vauum energy. Also a theoretical foundation of such an energy extraction is presented. It comprises the presence of an energy circulation in an electro- or magnetostatic field and the influence of electro- or magnetostatic fields on the propagation of the zero point (ground state) oszillations of the quantum electrodynamic vacuum.

These considerations suggest that the presence of a charge density $\rho(\vec{r})$ can be viewed as a result from a symmetry breaking, i.e. according to Eq. (17)

$$\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r}) = 0 \quad \Leftrightarrow \quad \phi(\vec{r}) = \sigma(\vec{r}) \qquad \text{for} \quad \rho(\vec{r}) = 0 \tag{31}$$

$$\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r}) \neq 0 \quad \Leftrightarrow \quad \phi(\vec{r}) \neq \sigma(\vec{r}) \qquad \text{for} \quad \rho(\vec{r}) \neq 0 \tag{32}$$

The considerations in this section indicate that new observable features can only be expected for more complex physical scenarios like electrostatics with a non-vanishing homogeneous current (see section 1.3), electro- and magnetostatics (see section 2 and 3) and electrodynamics.

1.2 The special case of a spin connection which reproduces the result of textbook electrostatics

In the previous section we have found that on the level of the potential $\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r})$ and field $\vec{E}\,(\vec{r})$ the textbook and ECE electrostatics are equivalent. There is another approach which likewise leads to a compatibility of ECE and textbook electrostatics. This is given by choosing a specific spin connection $\vec{\omega}\,(\vec{r})$ that results in a potential $\phi(\vec{r})$ and field $\vec{E}\,(\vec{r})$ which is equal to that of textbook electrostatics. This has already been reported in some ECE papers, see e.g. Ref. [15, 16]. For a point charge Q such a particular spin connection $\vec{\omega}\,(\vec{r})$ is, for example, given by

$$\vec{\omega} = \frac{\vec{r}}{r^2} \tag{33}$$

Let's verify that this spin connection $\vec{\omega}\,(\vec{r})$ reproduces the well-known Coloumb potential and field. By inserting the Coulomb potential of a point charge Q, i.e.

$$\phi = \frac{1}{4\,\pi\,\epsilon_0}\frac{Q}{r} \tag{34}$$

and the spin connection (33) in Eq. (3) we obtain its electric field:

$$\vec{E} = -\,\vec{\nabla}\,\phi + \vec{\omega}\,\phi = \frac{1}{4\,\pi\,\epsilon_0}\left(Q\,\frac{\vec{r}}{r^3} \,+\, Q\,\frac{\vec{r}}{r^3}\right) = \frac{2}{4\,\pi\,\epsilon_0}\,Q\,\frac{\vec{r}}{r^3} \tag{35}$$

According to Eq. (34) and (35) the redefined potential

$$\phi'(\vec{r}) = p_\phi\,\phi(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\frac{Q}{r} \quad \text{with} \quad p_\phi = 1 \tag{36}$$

and the redefined electric field

$$\vec{E}'\,(\vec{r}) = p_E\,\vec{E}\,(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,Q\,\frac{\vec{r}}{r^3} \quad \text{with} \quad p_E = \frac{1}{2} \tag{37}$$

reproduces the well-known Coulomb law. That means, from the ECE point of view, every charge is associated with a spin connection. Note, however, that the scaling factors p_ϕ and p_E of the potential $\phi(\vec{r})$ and electric field $\vec{E}\,(\vec{r})$, respectively, are not equal (which might be viewed as an unsatisfying situation):

$$p_\phi = 1 \;\neq\; p_E = \frac{1}{2} \tag{38}$$

From Eq. (9) we can infer that the scalar field $\sigma(\vec{r})$ which is associated with the potential $\phi(\vec{r})$, Eq. (34), and electric field $\vec{E}\,(\vec{r})$, Eq. (35), is given by

$$\sigma(\vec{r}) = -\frac{1}{4\,\pi\,\epsilon_0}\,\frac{Q}{r} = -\,\phi(\vec{r}) \tag{39}$$

Now let's consider the general case. As obvious from Eq. (16), a scalar field $\sigma(\vec{r})$ which has the same dependence on $\vec{r}$ as the integral term in Eq. (16) may reproduce the behavior of textbook electrostatics. That means if

$$\sigma(\vec{r}) = \frac{s}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\; d^3R \tag{40}$$

we obtain from Eq. (16)

$$\phi(\vec{r}) = \frac{1+s}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\; d^3R \tag{41}$$

whereby s is a parameter. The spin connection $\vec{\omega}\,(\vec{r})$, which is associated with the potentials (41) and (40), is obtained by inserting Eq. (40) into Eq. (18):

$$\vec{\omega}\,(\vec{r}) \;=\; -\,\frac{s}{1+s}\; \frac{\displaystyle\iiint \frac{(\vec{r} - \vec{R})\,\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|^3}\; d^3R}{\displaystyle\iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\; d^3R} \tag{42}$$

The electric field $\vec{E}\,(\vec{r})$ which is associated with the potentials (41) and (40) is obtained by inserting Eqs. (9), (40) and (41) into Eq. (3), namely

$$\vec{E}\,(\vec{r}) = -\,\vec{\nabla}\,(\phi - \sigma) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{(\vec{r} - \vec{R})\,\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|^{\,3}}\; d^3R \tag{43}$$

which is independent of the parameter s and equal to the electric field of textbook electrostatics like Eq. (19). According to Eq. (41) and (43) the redefined potential

$$\widetilde{\phi}(\vec{r}) = q_\phi\;\phi(\vec{r}) \quad \text{with} \quad q_\phi = \frac{1}{1+s} \tag{44}$$

and the redefined electric field

$$\widetilde{\vec{E}}(\vec{r}) = q_E\;\vec{E}\,(\vec{r}) \quad \text{with} \quad q_E = 1 \tag{45}$$

reproduces the result of textbook electrostatics. Note, however, that also here the scaling factors q_ϕ and q_E are not equal (which might be viewed as an unsatisfying situation):

$$q_\phi = \frac{1}{1+s} \neq q_E = 1 \tag{46}$$

Theoretically, the parameter s, and thus also the scaling factor q_ϕ , does not have to be constant. If it depends on $\vec{r}$, then the behavior of textbook electrostatics may be obtained by a spatially dependent redefinition of the potential.

We note that from the perspective of Eq. (16) and the considerations in this section, a deviation from textbook electrostatics appears if the potentials

$$\frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \qquad \text{and} \qquad \sigma(\vec{r}) \tag{47}$$

differ in their dependence on $\vec{r}$. However, the electrostatic ECE equations (1) and (2) do not specify the physical conditions which create such a difference.

1.3 Electrostatics in the presence of a homogeneous current

In this section we consider the electrostatic ECE equations for the case $\vec{j} \neq 0$ whereby $\vec{j} = \vec{j}\,(\vec{r})$ is the so-called homogeneous current. The homogeneous current $\vec{j}\,(\vec{r})$ is related to effects of gravitation on electromagnetism [10]. The physical circumstances under which a homogeneous current appears, or how it can be generated experimetally in the laboratory, are presently not clear and therefore it is usually assumed that $\vec{j} = 0$ [17]. Nevertheless, as will be shown in this section, the presence of a homogeneous current $\vec{j}\,(\vec{r})$ may create features in the potential $\phi(\vec{r})$ or $\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r})$ which are not known in textbook electrostatics. Thus, in contrast to the conclusions presented in section 1.1, the homogeneous current $\vec{j}\,(\vec{r})$ represents a possibility to create resonance-like features in the potential $\phi(\vec{r})$ or $\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r})$, at least theoretically.

In the case of $\vec{j} \neq 0$ the electrostatic ECE equations (1) $-$ (3) are modified with respect to the second equation, see e.g. Ref. [18], and the complete set is given by

$$\Delta\phi - \vec{\nabla} \cdot \left(\vec{\omega}\,\phi\right) = -\frac{\rho}{\epsilon_0} \tag{48}$$

$$\vec{\nabla} \times \left(\vec{\omega}\,\phi\right) = -\,\mu_0\,\vec{j} \tag{49}$$

$$\vec{E} \;=\; -\,\vec{\nabla}\phi \;+\; \vec{\omega}\,\phi \tag{50}$$

[10] In contrast to textbook physics the ECE Theory implies a coupling between gravitation and electromagnetism.

We assume that the homogeneous current $\vec{j}\,(\vec{r}\,)$ represents a given function like the charge density $\rho(\vec{r}\,)$. To find the general solution of Eqs. (48) and (49) we introduce a vector field $\vec{h}=\vec{h}\,(\vec{r}\,)$ by

$$\vec{j} = \vec{\nabla} \times \vec{h} \tag{51}$$

whereby $\vec{h}\,(\vec{r}\,)$ represents a vector potential of $\vec{j}\,(\vec{r}\,)$. This can be done because the left-hand side of Eq. (49) indicates that the homogeneous current $\vec{j}\,(\vec{r}\,)$ appears as a curl of a vector field. We recall that the vector potential $\vec{h}\,(\vec{r}\,)$ is an ambiguous function because the addition of a gradient of any scalar field leaves $\vec{j}\,(\vec{r}\,)$ unchanged. For a given vector field $\vec{j}\,(\vec{r}\,)$ there are many ways to find an associated vector potential $\vec{h}\,(\vec{r}\,)$, for example via the following formula [19]:

$$\vec{h}(\vec{r}\,) = -\,\vec{r} \times \left[\,\int_0^1 \vec{j}(s\,\vec{r}\,)\;s\;ds\right] \tag{52}$$

By inserting Eq. (51) in (49) we obtain

$$\vec{\nabla} \times \left(\vec{\omega}\;\phi\right) = -\,\mu_0\,\vec{\nabla} \times \vec{h} \tag{53}$$

By taking into account Eq. (8) we infer that Eq. (53) is satisfied if

$$\vec{\omega}\;\phi = -\,\mu_0\;\vec{h} + \vec{\nabla}\sigma \tag{54}$$

whereby $\sigma = \sigma(\vec{r}\,)$ represents an arbitrary scalar field. By inserting Eq. (54) in (48) we get

$$\Delta\phi - \vec{\nabla}\cdot\left(-\,\mu_0\;\vec{h} + \vec{\nabla}\sigma\right) = -\frac{\rho}{\epsilon_0} \tag{55}$$

and thus

$$\Delta(\phi - \sigma) = -\mu_0\;\vec{\nabla}\cdot\vec{h}\;\;-\;\;\frac{\rho}{\epsilon_0} \tag{56}$$

If $\vec{h}\,(\vec{r}\,)$ and $\rho(\vec{r}\,)$ are spatially limited, i.e. $\vec{h}\,(\vec{r}\,) \to 0$ for $r \to \infty$ and $\rho(\vec{r}\,) \to 0$ for $r \to \infty$, then the solution can be represented in analogy to Eqs. (11) $-$ (17), i.e. [11]

$$\phi(\vec{r}\,) = \frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{\rho(\vec{R}) + \epsilon_0\,\mu_0\,\vec{\nabla}\cdot\vec{h}\,(\vec{R})}{|\;\vec{r} - \vec{R}\;|}\;d^3R\;+\;\sigma(\vec{r}\,) \tag{57}$$

or

$$\beta(\vec{r}\,) = \phi(\vec{r}\,) - \sigma(\vec{r}\,) = \frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{\rho(\vec{R}) + \epsilon_0\,\mu_0\,\vec{\nabla}\cdot\vec{h}\,(\vec{R})}{|\;\vec{r} - \vec{R}\;|}\;d^3R \tag{58}$$

[11] For the sake of simplicity we have omitted in Eqs. (57) and (58) on the right-hand side the solution of the corresponding Laplace equation, see footnote 4 on page 7.

Thus the divergence term $\epsilon_0 \, \mu_0 \, \vec{\nabla} \cdot \vec{h}$ corresponds to a charge density, at least formally. By inserting Eq. (57) into Eq. (54) as well as Eqs. (57), (58) and (54) into Eq. (50) we obtain the spin connection $\vec{\omega}(\vec{r})$ and the electric field $\vec{E}(\vec{r})$ which are associated with the potential (57), namely

$$\vec{\omega}(\vec{r}) = \frac{\vec{\nabla}\sigma(\vec{r}) \; - \; \mu_0 \, \vec{h}(\vec{r})}{\displaystyle\iiint \frac{\rho(\vec{R}) + \epsilon_0 \, \mu_0 \, \vec{\nabla} \cdot \vec{h}(\vec{R})}{4 \, \pi \, \epsilon_0 \, | \, \vec{r} \, - \, \vec{R} \, |} \, d^3 R \; + \; \sigma(\vec{r})} \tag{59}$$

$$\begin{aligned} \vec{E}(\vec{r}) &= -\mu_0 \, \vec{h}(\vec{r}) - \vec{\nabla}\beta(\vec{r}) \\ &= -\mu_0 \, \vec{h}(\vec{r}) \; + \iiint \frac{(\vec{r} \, - \, \vec{R}) \left[\rho(\vec{R}) + \epsilon_0 \, \mu_0 \, \vec{\nabla} \cdot \vec{h}(\vec{R}) \right]}{4 \, \pi \, \epsilon_0 \, | \, \vec{r} \, - \, \vec{R} \, |^3} \, d^3 R \end{aligned} \tag{60}$$

Like that electric field which is represented by Eq. (19), the electric field given by Eq. (60) does not depend on the arbitrary scalar field $\sigma(\vec{r})$. According to Eqs. (56) – (60) the vector field $\vec{j}(\vec{r}) = \vec{\nabla} \times \vec{h}(\vec{r})$ may lead to modifications of the potential $\phi(\vec{r})$ or $\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r})$ and field $\vec{E}(\vec{r})$. Such modifications are not known in textbook electrostatics. For an appropriate spatial dependence the homogeneous current $\vec{j}(\vec{r}) = \vec{\nabla} \times \vec{h}(\vec{r})$ may create resonance-like features in the potential $\phi(\vec{r})$ or $\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r})$.

According to Eq. (58) the case of $\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r}) \neq 0$ may be viewed as a breaking of a symmetry:

$$\begin{aligned} \beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r}) &= 0 \\ \text{for charge density} \left[\rho(\vec{r}) + \epsilon_0 \, \mu_0 \, \vec{\nabla} \cdot \vec{h}(\vec{r}) \right] &= 0 \end{aligned} \tag{61}$$

$$\begin{aligned} \beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r}) &\neq 0 \\ \text{for charge density} \left[\rho(\vec{r}) + \epsilon_0 \, \mu_0 \, \vec{\nabla} \cdot \vec{h}(\vec{r}) \right] &\neq 0 \end{aligned} \tag{62}$$

We note that, according to Eqs. (49) – (60), the vector field $\vec{h}(\vec{r})$ should be a source and rotational field at the same time, i.e.

$$\vec{\nabla} \cdot \vec{h} \; \neq \; 0 \qquad \text{and} \qquad \vec{\nabla} \times \vec{h} \; \neq \; 0 \tag{63}$$

Otherwise there is no difference to the case described in section 1.1. For comparison we note that in textbook electro- and magnetostatics the electric field $\vec{E}$ is a non-rotational source field. i.e.

$$\vec{\nabla} \cdot \vec{E} \neq 0 \quad \text{and} \quad \vec{\nabla} \times \vec{E} = 0 \tag{64}$$

and the magnetic field $\vec{B}$ is a source-free rotational field, i.e.

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{B} \neq 0 \tag{65}$$

1.4 Summary

By introducing two further scalar potentials, $\sigma(\vec{r})$ and $\beta(\vec{r})$, by

$$\vec{\omega}\,\phi = \vec{\nabla}\sigma \tag{66}$$

$$\beta = \phi - \sigma \tag{67}$$

the electrostatic ECE equations (1) $-$ (3) were transformed into

$$\Delta\,\beta = -\frac{\rho}{\epsilon_0} \tag{68}$$

$$\vec{E} = -\,\vec{\nabla}\beta \tag{69}$$

These equations do not contain the vector spin connection $\vec{\omega}$ any more and permit a direct comparison with the equations of textbook electrostatics. Actually, with respect to the scalar potential β and electric field $\vec{E}$ the Eqs. (68) and (69) are equivalent to the equations of textbook electrostatics. For a spatially limited charge density ρ the solution of Eq. (68) is given by [12]

$$\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \tag{70}$$

and thus

$$\vec{E}\,(\vec{r}) = \vec{E}_\phi\,(\vec{r}) - \vec{E}_\sigma\,(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{(\vec{r} - \vec{R})\,\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|^3}\, d^3R \tag{71}$$

whereby

$$\vec{E}_\phi = -\,\vec{\nabla}\phi \tag{72}$$

[12] The most general solution of Eq. (68) is given by
$\beta(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \;+\; \beta_{hom}(\vec{r})$
whereby $\beta_{hom}(\vec{r})$ is a solution of the Laplace equation $\Delta\beta_{hom} = 0$

$$\vec{E}_\sigma = -\vec{\nabla}\sigma \tag{73}$$

Thus, on the level of the potential $\beta(\vec{r})$ and field $\vec{E}(\vec{r})$ the textbook and ECE electrostatics are equivalent. In ECE electrostatics, however, the potential and field arises from a difference of two potentials or fields which both depend on $\vec{r}$. This peculiar feature is an effect of the vector spin connection $\vec{\omega}(\vec{r})$ which is related to the spinning or torsion of space-time. We emphasize that the potential $\phi(\vec{r})$ which appears in the electrostatic ECE equations is not identical with the potential $\beta(\vec{r})$ of textbook electrostatics. In contrast to that, the field $\vec{E}(\vec{r})$ which appears in the electrostatic ECE equations is equal to the field of textbook electrostatics.

The potentials $\phi(\vec{r})$ and $\sigma(\vec{r})$ and fields $\vec{E}_\phi(\vec{r})$ and $\vec{E}_\sigma(\vec{r})$, which are not explicitly specified by the electrostatic ECE equations, point to a possible existence of a physical reality beyond that of textbook electrostatics. We raise the question if these potentials and fields, or their effects, are physically measurable and if they carry the same physical characteristics like those of $\beta(\vec{r})$ and $\vec{E}(\vec{r})$.

Another significant question is which of the three associated potentials $\beta(\vec{r})$, $\phi(\vec{r})$ and $\sigma(\vec{r})$ and which of the three associated fields $\vec{E}(\vec{r})$, $\vec{E}_\phi(\vec{r})$ and $\vec{E}_\sigma(\vec{r})$ is the relevant quantity (in a specific context or problem). It seems that there are essentially two answers or views:

1. The relevant quantity is $\phi(\vec{r})$ or $\sigma(\vec{r})$ as well as $\vec{E}_\phi(\vec{r})$ or $\vec{E}_\sigma(\vec{r})$. These quantities can be considered as a modification of the corresponding textbook quantity, e.g. $\phi(\vec{r}) = \beta(\vec{r}) + \sigma(\vec{r})$ represents a modification of $\beta(\vec{r})$ which is the potential of textbook electrostatics. This view predicts a potential and field which is always different from that of textbook electrostatics. This might appear unsatisfying, however it is conceivable that the additional quantity such as $\sigma(\vec{r})$ develops its efficacy only under special conditions.

2. The relevant quantity is the potential $\beta(\vec{r})$ and field $\vec{E}(\vec{r})$ of textbook electrostatics, whereas the potentials $\phi(\vec{r})$ and $\sigma(\vec{r})$ and fields $\vec{E}_\phi(\vec{r})$ and $\vec{E}_\sigma(\vec{r})$ belong to a more subtle physical reality which is experimentally not yet explored.

The potential $\sigma(\vec{r})$, however, is a nearly arbitrary function, i.e. it is not explicitly specified by the electrostatic ECE equations, and thus also the potential $\phi(\vec{r})$ is an ambiguous quantity. Therefore the first view seems to be unsatisfying and we conclude that the second view makes more sense. This means that ECE electrostatics in the form of Eqs. (1) $-$ (3) does not imply new phenomena which are (easily) verifiable in an experimental manner, even if any electrical charge is associated with a vector spin connection $\vec{\omega}(\vec{r})$ and the potential $\beta(\vec{r})$ and field $\vec{E}(\vec{r})$ emerge from a difference of two potentials or fields which both depend on $\vec{r}$.
However, if we assume that the so-called homogeneous current $\vec{j}(\vec{r})$ is not zero, then the electrostatic potentials and fields may show novel features which are not possible

in textbook electrostatics. According to ECE Theory the homogeneous current $\vec{j}\,(\vec{r})$ is related to effects of gravitation on electromagnetism. The electrostatic ECE equations for $\vec{j}\,(\vec{r}) \neq 0$ are given by Eqs. (48) − (50). In a similar way as above and by introducing the vector potential $\vec{h}\,(\vec{r})$ of the homogeneous current $\vec{j}\,(\vec{r})$ by

$$\vec{j} = \vec{\nabla} \times \vec{h} \tag{74}$$

the Eqs. (48) − (50) can be transformed into equations which do not comprise the vector spin connection $\vec{\omega}\,(\vec{r})$ any more. The resulting potential $\beta(\vec{r})$ and field $\vec{E}\,(\vec{r})$ is given by [13]

$$\beta(\vec{r}) = \phi(\vec{r}) - \sigma(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R}) + \epsilon_0\,\mu_0\,\vec{\nabla}\cdot\vec{h}\,(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \tag{75}$$

$$\begin{aligned} \vec{E}\,(\vec{r}) &= -\mu_0\,\vec{h}\,(\vec{r}) - \vec{\nabla}\beta(\vec{r}) \\ &= -\mu_0\,\vec{h}\,(\vec{r}) \;+ \iiint \frac{(\vec{r} - \vec{R})\left[\rho(\vec{R}) + \epsilon_0\,\mu_0\,\vec{\nabla}\cdot\vec{h}\,(\vec{R})\right]}{4\,\pi\,\epsilon_0\,|\,\vec{r} - \vec{R}\,|^3}\, d^3R \end{aligned} \tag{76}$$

Even if it is presently not clear how to generate a homogeneous current experimetally in the laboratory, its influence on electrostatic potentials and fields is an interesting effect, at least theoretically.

2 The former set of the electro- and magnetostatic ECE equations

In this section we will consider the former set of electro- and magnetostatic ECE equations which are eight equations with eight variables. We will show that these equations can be transformed into 4 equations with 4 variables. This can be done by the introduction of a scalar field $g(\vec{r})$, similar to the scalar field $\sigma(\vec{r})$ used in section 1. This scalar field $g(\vec{r})$ stands for the difference between ECE and textbook electrostatics and in this sense it assumes the role of the overall four spin connection components $\vec{\omega}\,(\vec{r})$ and $\omega_0(\vec{r})$.

2.1 The set of equations in its original form

The former set of the electro- and magnetostatic ECE equations in vector notation, see e.g. Ref. [20] or [21], is given by [14]

[13] For the sake of simplicity we have omitted in Eq. (75) on the right-hand side the solution of the corresponding Laplace equation, see footnote 12 on page 17.

[14] Eqs. (77) − (82) refer to the assumption that the so-called polarization index can be omitted, i.e. one polarization only, see e.g. Ref. [21].

Field equations in terms of potentials:

$$\vec{\nabla} \cdot \left(\vec{\omega} \times \vec{A} \right) = 0 \tag{77}$$

$$\vec{\nabla} \times \left(\vec{\omega}\, \phi - \vec{A}\, \omega_0 \right) = 0 \tag{78}$$

$$\Delta \phi - \vec{\nabla} \cdot \left(\vec{\omega}\, \phi - \vec{A}\, \omega_0 \right) = -\frac{\rho}{\epsilon_0} \tag{79}$$

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} - \vec{\omega} \times \vec{A} \right) &= \mu_0\, \vec{J} \\ &\text{or} \\ \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) \; - \; \Delta \vec{A} \; - \; \vec{\nabla} \times \left(\vec{\omega} \times \vec{A} \right) &= \mu_0\, \vec{J} \end{aligned} \tag{80}$$

Field-potential relations:

$$\vec{E} = -\, \vec{\nabla} \phi + \vec{\omega}\, \phi - \vec{A}\, \omega_0 \tag{81}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \; - \; \vec{\omega} \times \vec{A} \tag{82}$$

whereby $\vec{A} = \vec{A}(\vec{r})$ is the vector potential, $\vec{J} = \vec{J}(\vec{r})$ the current density, $\omega_0 = \omega_0(\vec{r})$ the so-called scalar spin connection, $\vec{\omega} = \vec{\omega}(\vec{r})$ the so-called vector spin connection, $\vec{E} = \vec{E}(\vec{r})$ the electric field, and $\vec{B} = \vec{B}(\vec{r})$ the magnetic field. The two different forms of equation Eq. (80) are based on the relation

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} \; = \; \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) \; - \; \Delta \vec{A} \tag{83}$$

Eqs. (77) – (82) merge into the equations of textbook electro- and magnetostatics if

- $\vec{\omega}\, \phi = \vec{A}\, \omega_0$

  which implies $\vec{\omega} \times \vec{A} = 0$ because in this case $\vec{\omega}$ and $\vec{A}$ are (anti)parallel to each other – this condition is reported in Ref [13] for the electrodynamic equations and works also for the electro- and magnetostatic case

  or

- $\vec{\omega} = 0$ and $\omega_0 = 0$

2.2 Results and discussion of the transformed equations

Eqs. (77) – (80) represent 8 equations to determine the 8 quantities ω_0 , $\vec{\omega}$, ϕ and $\vec{A}$. Nevertheless, due to the presence of rotational fields and operators, the solutions comprise an ambiguousness. By taking into account Eq. (8) we may infer from Eq. (78) that

$$\vec{\omega}\,\phi - \vec{A}\,\omega_0 = \vec{\nabla} g \tag{84}$$

whereby $g = g(\vec{r})$ represents a scalar field which is apriori ambiguous. Now let's take the cross product with $\vec{A}$ on both sides of Eq. (84), i.e.

$$\left(\vec{\omega}\,\phi - \vec{A}\,\omega_0\right) \times \vec{A} = \left(\vec{\nabla} g\right) \times \vec{A} \tag{85}$$

Because the second term on the left side vanishes we get

$$\vec{\omega} \times \vec{A} = \frac{\left(\vec{\nabla} g\right) \times \vec{A}}{\phi} \tag{86}$$

By inserting Eq. (86) into Eqs. (77) and (80) as well as Eq. (84) into Eq. (79) we obtain

$$\vec{\nabla} \cdot \left[\frac{\left(\vec{\nabla} g\right) \times \vec{A}}{\phi} \right] = 0 \tag{87}$$

$$-\,\vec{\nabla} \times \left[\frac{\left(\vec{\nabla} g\right) \times \vec{A}}{\phi} \right] + \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0\,\vec{J} \tag{88}$$

$$\Delta(\phi - g) = \Delta\,\beta = -\frac{\rho}{\epsilon_0} \tag{89}$$

whereby

$$\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) \tag{90}$$

Thus we have reduced the eight Eqs. (77) – (80) with eight variables to the five equations (87) – (89) with five variables ϕ, g and $\vec{A}$. By inserting Eq. (84) in (81) and (86) in (82) we get the corresponding electric and magnetic field:

$$\vec{E} = -\,\vec{\nabla}\,(\phi - g) = -\,\vec{\nabla}\beta \tag{91}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} - \left[\frac{\left(\vec{\nabla} g\right) \times \vec{A}}{\phi} \right] \tag{92}$$

Interestingly, looking at Eqs. (87) – (92), the spin connections $\vec{\omega}(\vec{r})$ and $\omega_0(\vec{r})$ do not appear any more. Thus, in this approach or representation the difference between ECE and textbook electro- and magnetostatics is given by a single quantity, namely the scalar field $g(\vec{r})$ which represents a potential. This resembles to the approach presented in section 1 where the related potential $\sigma(\vec{r})$ takes quasi over the role of $\vec{\omega}(\vec{r})$.

For a spatially limited charge density $\rho(\vec{r})$, i.e. $\rho(\vec{r}) \to 0$ for $r \to \infty$, the well-known solution of Eq. (89) is given by [15]

$$\boxed{\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r}\,-\,\vec{R}\,|}\; d^3R \qquad (93)}$$

By inserting Eq. (93) into Eq. (91) the electric field $\vec{E}$ becomes

$$\begin{aligned} \vec{E}(\vec{r}) &= \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r}) = -\,\vec{\nabla}\left[\phi(\vec{r}) - g(\vec{r})\right] = -\,\vec{\nabla}\beta(\vec{r}) \\ &= \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{(\vec{r}\,-\,\vec{R})\,\rho(\vec{R})}{|\,\vec{r}\,-\,\vec{R}\,|^3}\; d^3R \end{aligned} \qquad (94)$$

whereby we have introduced the electric fields $\vec{E}_\phi$ and $\vec{E}_g$ by

$$\vec{E}_\phi = -\,\vec{\nabla}\phi \qquad (95)$$

$$\vec{E}_g = -\,\vec{\nabla} g \qquad (96)$$

According to Eq. (93) and (94) the scalar potential $\beta(\vec{r})$ and electric field $\vec{E}(\vec{r})$ is equal to the scalar potential and electric field of textbook electro- and magnetostatics, respectively. Thus, on the level of the scalar potential $\beta(\vec{r})$ and electric field $\vec{E}(\vec{r})$ the ECE and textbook electro- and magnetostatics are equivalent. However, in ECE Theory the scalar potential $\beta(\vec{r})$ and electric field $\vec{E}(\vec{r})$ emerge from a difference between two quantities which both depend on $\vec{r}$, whereas in textbook electro- and magnetostatics they result always from one spatially dependent function.

According to Eqs. (93) – (96) the case of

$$g(\vec{r}) = g_0 \qquad (97)$$

, whereby g_0 is a constant, implies

$$\beta(\vec{r}) = \phi(\vec{r}) - g_0 \qquad (98)$$

[15] The most general solution of Eq. (89) is given by
$\beta(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r}\,-\,\vec{R}\,|}\; d^3R \;+\; \beta_{hom}(\vec{r})$
whereby $\beta_{hom}(\vec{r})$ is a solution of the Laplace equation $\Delta\beta_{hom} = 0$

$$\vec{E}_g(\vec{r}) = 0 \tag{99}$$

$$\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) = -\vec{\nabla}\beta(\vec{r}) \tag{100}$$

and corresponds to the scenario of textbook electro- and magnetostatics.

We emphasize that ECE and textbook electro- and magnetostatics usually use the same symbol ϕ for the scalar potential. However, according to the above-mentioned results they do not represent the same quantity and have to be distinguished. In this paper ϕ represents the scalar potential which appears in the original electro- and magnetostatic ECE equations (77) − (82) and β corresponds to the scalar potential of textbook electro- and magnetostatics. In contrast to that, the electric field $\vec{E}$, which likewise appears in the original electro- and magnetostatic ECE equations (77) − (82), is identical to the electric field of textbook electro- and magnetostatics.

Now let's consider the magnetic field $\vec{B}$. By inserting Eq. (82) into the upper part of Eq. (80) or by inserting Eq. (92) into Eq. (88) we obtain

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{101}$$

which is the same relation between magnetic field $\vec{B}$ and current density $\vec{J}$ as in textbook magneto- and electrostatics. From Eq. (77) [16] we infer that $\vec{\omega} \times \vec{A}$ can be written as

$$\vec{\omega} \times \vec{A} = \vec{\nabla} \times \vec{V} \tag{102}$$

whereby $\vec{V} = \vec{V}(\vec{r})$ is another vector potential which depends in some way on the vector poential $\vec{A} = \vec{A}(\vec{r})$. We introduce another vector potential $\vec{\Lambda} = \vec{\Lambda}(\vec{r})$ by

$$\vec{\Lambda} = \vec{A} - \vec{V} \tag{103}$$

From Eqs. (82), (102), and (103) it follows that the magnetic field $\vec{B}$ can be represented as

$$\begin{aligned} \vec{B} \; &= \; \vec{\nabla} \times \vec{\Lambda} \; = \; \vec{\nabla} \times \left(\vec{A} - \vec{V}\right) \; = \; \vec{\nabla} \times \vec{A} - \vec{\nabla} \times \vec{V} \\ &= \; \vec{\nabla} \times \vec{A} - \vec{\omega} \times \vec{A} \end{aligned} \tag{104}$$

With respect to the magnetic field $\vec{B}$, current density $\vec{J}$, and vector potential $\vec{\Lambda}$, the Eqs. (101) and (104) are the same as those in textbook magneto- and electrostatics. Thus $\vec{\Lambda}$ coresponds to the vector potential used in textbook magneto- and electrostatics. By inserting Eq. (104) into Eq. (101) and by means of Eq. (83) we get

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{\Lambda}\right) \; = \; \vec{\nabla}\left(\vec{\nabla} \cdot \vec{\Lambda}\right) - \Delta \vec{\Lambda} \; = \; \mu_0 \vec{J} \tag{105}$$

[16] Looking at Eqs. (82) and (77) we see that the latter is equivalent to $\vec{\nabla} \cdot \vec{B} = 0$ which means the absence of magnetic monopoles.

By using the so-called Coulomb gauge $\vec{\nabla} \cdot \vec{\Lambda} = 0$, see footnote 29 in section 3, Eq. (105) becomes

$$\Delta \vec{\Lambda} = - \mu_0 \vec{J} \tag{106}$$

For a spatially limited current density $\vec{J}(\vec{r})$, i.e. $\vec{J}(\vec{r}) \to 0$ for $r \to \infty$, the vector potential $\vec{\Lambda} = \vec{A} - \vec{V}$ which solves Eq. (106) is well-known, namely [17]

$$\boxed{\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r})}{|\vec{r} - \vec{R}|} \, d^3R} \tag{107}$$

By inserting Eq. (107) into Eq. (104) the magnetic field $\vec{B}$ results in

$$\begin{aligned} \vec{B}(\vec{r}) &= \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r}) = \vec{\nabla} \times \left[\vec{A}(\vec{r}) - \vec{V}(\vec{r}) \right] = \vec{\nabla} \times \vec{\Lambda}(\vec{r}) \\ &= \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{R}) \times (\vec{r} - \vec{R})}{|\vec{r} - \vec{R}|^3} \, d^3R \end{aligned} \tag{108}$$

whereby we have introduced the magnetic fields $\vec{B}_A$ and $\vec{B}_V$ by

$$\vec{B}_A = \vec{\nabla} \times \vec{A} \tag{109}$$

$$\vec{B}_V = \vec{\nabla} \times \vec{V} = \vec{\omega} \times \vec{A} \tag{110}$$

According to Eq. (107) and (108) the vector potential $\vec{\Lambda}(\vec{r})$ and magnetic field $\vec{B}(\vec{r})$ is equal to the vector potential and magnetic field of textbook magneto- and electrostatics, respectively. Thus, on the level of the vector potential $\vec{\Lambda}(\vec{r})$ and magnetic field $\vec{B}(\vec{r})$ textbook and ECE magneto- and electrostatics are equivalent. However, in ECE Theory

- the vector potential $\vec{\Lambda}(\vec{r})$ emerges from a difference between two vector potentials whose curl do not vanish
- the magnetic field $\vec{B}(\vec{r})$ emerges from a difference between two magnetic fields which both depend on $\vec{r}$

whereas in textbook electro- and magnetostatics they result always from one spatially vector field.

[17] The most general solution of Eq. (106) is given by
$\vec{\Lambda}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{R})}{|\vec{r} - \vec{R}|} \, d^3R + \vec{\Lambda}_{hom}(\vec{r})$
whereby the components of $\vec{\Lambda}_{hom}(\vec{r})$ are solutions of the Laplace equations $\Delta \vec{\Lambda}_{hom} = 0$

According to Eqs. (103), (104), (108) − (110) the case of

$$\vec{V}(\vec{r}) = \vec{\nabla}\alpha(\vec{r}) \tag{111}$$

, whereby $\alpha(\vec{r})$ is any scalar field, implies

$$\vec{\nabla} \times \vec{V}(\vec{r}) = 0 \tag{112}$$

$$\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{\nabla}\alpha(\vec{r}) \tag{113}$$

$$\vec{B}_V(\vec{r}) = 0 \tag{114}$$

$$\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = \vec{\nabla} \times \vec{\Lambda}(\vec{r}) \tag{115}$$

and corresponds to the scenario of textbook magneto- and electrostatics. Another way to describe this case is obvious from Eqs. (87) − (92), namely by $g(\vec{r}) = g_0$ whereby g_0 is constant.

We emphasize that ECE and textbook magneto- and electrostatics usually use the same symbol $\vec{A}$ for the vector potential. However, according to the above-mentioned results they do not represent the same quantity and have to be distinguished. In this paper $\vec{A}$ represents the vector potential which appears in the original magneto- and electrostatic ECE equations (77) − (82) and $\vec{\Lambda}$ corresponds to the vector potential of textbook magneto- and electrostatics. In contrast to that, the magnetic field $\vec{B}$, which likewise appears in the original magneto- and electrostatic ECE equations (77) − (82), is identical to the magnetic field of textbook magneto- and electrostatics.

The above considerations have shown the equivalence of ECE and textbook electro- and magnetostatics on the level of the

- scalar potential $\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r})$
- electric field $\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r})$
- vector potential $\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r})$
- magnetic field $\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r})$

This means that the scalar potential $\beta(\vec{r})$, electric field $\vec{E}(\vec{r})$, vector potential $\vec{\Lambda}(\vec{r})$ and magnetic field $\vec{B}(\vec{r})$ is identical with the scalar potential, electric field, vector potential and magnetic field of textbook electro- and magnetostatics, respectively. In ECE electro- and magnetostatics, however, these four quantities (we call them level I quantities) emerge from a difference of two quantities (we call them level II quantities) which both depend on $\vec{r}$, whereas in textbook electro- and magnetostatics they result always from one spatially dependent scalar or vector field which corresponds to the case $g(\vec{r}) = \textit{constant}$. The level II scalar potentials $\phi(\vec{r})$ and $g(\vec{r})$, level II electric fields $\vec{E}_\phi(\vec{r})$ and $\vec{E}_g(\vec{r})$, level II vector potentials $\vec{A}(\vec{r})$

and $\vec{V}(\vec{r})$, and level II magnetic fields $\vec{B}_A(\vec{r})$ and $\vec{B}_V(\vec{r})$ point to the existence of a possible physical reality beyond that of textbook electro- and magnetostatics. We raise the following questions:

- Are the level II potentials and fields, or their effects, physically detectable ? Possibly their experimental verification requires special circumstances and / or special measurement techniques.
- Do the level II potentials and fields carry the same physical characteristics like those of the level I quantities $\beta(\vec{r})$, $\vec{E}(\vec{r})$, $\vec{\Lambda}(\vec{r})$ and $\vec{B}(\vec{r})$?
- Which of three related quantities such as $\beta(\vec{r})$, $\phi(\vec{r})$ and $g(\vec{r})$ is the relevant quantity ? Possibly this depends on the specific context or problem.

Concerning the latter question it seems that there are two possibilities how to view the physical meaning of three related quantities such as $\beta(\vec{r})$, $\phi(\vec{r})$ and $g(\vec{r})$:

1. The relevant quantity is that which corresponds to that of textbook electro- and magnetostatics, i.e. for example $\beta(\vec{r})$, whereas the level II quantities such as $\phi(\vec{r})$ and $g(\vec{r})$ belong to a more subtle physical reality which is experimentally not yet explored.
2. The relevant quantity is one of the two level II quantities, i.e. for example $\phi(\vec{r})$ or $g(\vec{r})$, and represents a modification of the quantity of textbook electro- and magnetostatics. For example, $\phi(\vec{r}) = \beta(\vec{r}) + g(\vec{r})$ represents a modification of $\beta(\vec{r})$.

On the level of the four quantities $\beta(\vec{r})$, $\vec{E}(\vec{r})$, $\vec{\Lambda}(\vec{r})$ and $\vec{B}(\vec{r})$ textbook and ECE electro- and magnetostatics are equivalent. The potentials $\beta(\vec{r})$ and $\vec{\Lambda}(\vec{r})$ are given by Eqs. (93) and (107) which are solutions of the linear decoupled second-order differential equations (89) and (106). The fact that Eqs. (89) and (106) are decoupled means that there is no coupling between electric and magnetic quantities. This is obvious from Eqs. (93), (94), (107) and (108) which show that

- the scalar potential $\beta(\vec{r})$ and electric field $\vec{E}(\vec{r})$ is exclusively specified by the charge density $\rho(\vec{r})$
- the vector potential $\vec{\Lambda}(\vec{r})$ and magnetic field $\vec{B}(\vec{r})$ is exclusively specified by the current density $\vec{J}(\vec{r})$

Interestingly, however, for the level II potentials and fields $\phi(\vec{r})$, $g(\vec{r})$, $\vec{E}_\phi(\vec{r})$, $\vec{E}_g(\vec{r})$, $\vec{A}(\vec{r})$, $\vec{V}(\vec{r})$, $\vec{B}_A(\vec{r})$ and $\vec{B}_V(\vec{r})$ there is a coupling between electric and magnetic quantities, i.e. in general each of these quantities depends on $\rho(\vec{r})$ and $\vec{J}(\vec{r})$. This coupling is described by Eqs. (87) $-$ (89). Once the scalar potentials $\phi(\vec{r})$ and $g(\vec{r})$ and the vector potential $\vec{A}(\vec{r})$ are determined from Eqs. (87) $-$ (89), the other level II quantities can be computed from Eqs. (95), (96) and (107) $-$ (110).

Now we turn once again to Eqs. (87) – (89). In contrast to the (nearly) arbitrary scalar potential $\sigma(\vec{r})$ used in section 1, the related scalar potential $g(\vec{r})$ in Eqs. (87) – (89) appears apriori not as an ambiguous quantity. Nevertheless, the structure of Eqs. (87) – (89) suggests the existence of an ambiguousness in the overall solution. For example, an ambiguousness in the overall solution of Eqs. (87) – (89) may arise from the transformation

$$\frac{\left(\vec{\nabla} g\right) \times \vec{A}}{\phi} \;\rightarrow\; \frac{\left(\vec{\nabla} g\right) \times \vec{A}}{\phi} + \vec{\nabla} w \tag{116}$$

whereby $w = w(\vec{r})$ is a scalar field. This transformation leaves Eq. (88) unchanged, whereas in Eq. (87) an additional term Δw appears. However, if the scalar field $w = w(\vec{r})$ satisfies the Laplace equation, i.e. $\Delta\, w = 0$, then also Eq. (87) remains unchanged.

In Eq. (84), with regard to Eq. (78), the scalar potential $g(\vec{r})$ appears as an ambiguous quantity. It seems as if the ambiguousness of $g(\vec{r})$, when going from Eq. (84) to Eqs. (87) – (89), is shifted to the spin connection which is, however, not needed any more for the determination of $\phi(\vec{r})$ and $\vec{A}(\vec{r})$. According to Eq. (84) the vector spin connection $\vec{\omega}(\vec{r})$ is given by

$$\vec{\omega} = \frac{\vec{\nabla} g + \vec{A}\,\omega_0}{\phi} \tag{117}$$

whereby the scalar spin connection $\omega_0(\vec{r})$ is undetermined and represents an arbitrary function.

2.3 The equations for the level II potentials

Eqs. (87) – (89) can be further simplified to four coupled first order differential equations with four variables in the following way. By inserting Eq. (101) into Eq. (88) we get

$$-\vec{\nabla} \times \left[\frac{\left(\vec{\nabla} g\right) \times \vec{A}}{\phi} + \vec{\nabla} \times \vec{A}\right] = \vec{\nabla} \times \vec{B} \tag{118}$$

By taking into account Eq. (8) we infer that Eq. (118) is satisfied when

$$-\frac{\left(\vec{\nabla} g\right) \times \vec{A}}{\phi} + \vec{\nabla} \times \vec{A} = \vec{B} + \vec{\nabla}\psi \tag{119}$$

whereby $\psi = \psi(\vec{r})$ is apriori an arbitrary scalar field. From Eq. (90) we obtain

$$\vec{\nabla} g = \vec{\nabla}\phi - \vec{\nabla}\beta \tag{120}$$

Inserting Eq. (94) into Eq. (120) yields

$$\vec{\nabla} g = \vec{\nabla}\phi + \vec{E} \tag{121}$$

By inserting Eq. (121) into Eqs. (87) and (119) we get

$$\vec{\nabla} \cdot \left[\frac{\vec{E} + \vec{\nabla}\phi}{\phi} \times \vec{A} \right] = 0 \tag{122}$$

$$-\frac{\vec{E} + \vec{\nabla}\phi}{\phi} \times \vec{A} + \vec{\nabla} \times \vec{A} = \vec{B} + \vec{\nabla}\psi \tag{123}$$

$$\Delta \psi = 0 \tag{124}$$

whereby the level I fields, which are given by Eqs. (94) and (108),

$$\vec{E}(\vec{r}) = -\vec{\nabla}\beta(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{R})\,(\vec{r} - \vec{R})}{|\,\vec{r} - \vec{R}\,|^3}\, d^3R \tag{125}$$

$$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{\Lambda}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{R}) \times (\vec{r} - \vec{R})}{|\,\vec{r} - \vec{R}\,|^3}\, d^3R \tag{126}$$

are considered as given functions because they depend exclusively on the charge density $\rho(\vec{r})$ or current density $\vec{J}(\vec{r})$. The scalar field $\psi(\vec{r})$ represents the presence of an ambiguousness in the solutions.

Eq. (124) comes about by taking the divergence of Eq. (123) and taking into account Eq. (122) and $\vec{\nabla} \cdot \vec{B} = 0$. The latter is obvious from Eq. (126) and means the absence of magnetic monopoles.

Thus we have obtained four coupled first-order differential equations, namely Eqs. (122) and (123), with four variables $\phi(\vec{r})$ and $\vec{A}(\vec{r})$. Once $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ are determined from Eqs. (122) − (126), their associated quantities $g(\vec{r})$, $\vec{E}_\phi(\vec{r})$, $\vec{E}_g(\vec{r})$, $\vec{V}(\vec{r})$, $\vec{B}_A(\vec{r})$ and $\vec{B}_V(\vec{r})$ can be computed from Eqs. (93), (95), (96), (107), (109) and (110).

Let's turn to the statement that Eqs. (122) and (123) are first-order differential equations. Obviously, Eq. (123) comprises only first order derivatives. This is also true, but maybe less obvious, for Eq. (122). To verify this we evaluate the divergence term of Eq. (122) according to the well-known rules of vector calculus [18]:

$$\vec{\nabla} \cdot \left[\frac{\vec{E} + \vec{\nabla}\phi}{\phi} \times \vec{A} \right] = \vec{A} \cdot \left(\vec{\nabla} \times \frac{\vec{E} + \vec{\nabla}\phi}{\phi} \right) - \left(\frac{\vec{E} + \vec{\nabla}\phi}{\phi} \right) \cdot \left(\vec{\nabla} \times \vec{A} \right) \tag{127}$$

The term in the first bracket on the right-hand side yields

$$\vec{\nabla} \times \frac{\vec{E} + \vec{\nabla}\phi}{\phi} = \vec{\nabla} \times \frac{\vec{E}}{\phi} + \vec{\nabla} \times \left[\vec{\nabla} \ln\left(\frac{\phi}{\phi_0} \right) \right] \tag{128}$$

[18] Examples of useful relations are $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$ and $\vec{\nabla} \times (s\,\vec{a}) = s\,(\vec{\nabla} \times \vec{a}) + (\vec{\nabla} s) \times \vec{a}$ whereby $\vec{a} = \vec{a}(\vec{r})$ and $\vec{b} = \vec{b}(\vec{r})$ are any vector fields and $s = s(\vec{r})$ is any scalar field.

whereby ϕ_0 is a constant. Because the second term on the right-hand side vanishes we obtain

$$\vec{\nabla} \times \frac{\vec{E} + \vec{\nabla}\phi}{\phi} \;=\; \vec{\nabla} \times \frac{\vec{E}}{\phi} \;=\; \frac{1}{\phi}\,\vec{\nabla} \times \vec{E} \;+\; \left[\vec{\nabla}\left(\frac{1}{\phi}\right)\right] \times \vec{E} \tag{129}$$

According to Eq. (125) the electric field $\vec{E}$ is curl-free, i.e. $\vec{\nabla} \times \vec{E} = 0$ and we get

$$\vec{\nabla} \times \frac{\vec{E} + \vec{\nabla}\phi}{\phi} \;=\; \left[\vec{\nabla}\left(\frac{1}{\phi}\right)\right] \times \vec{E} = -\frac{1}{\phi^2}\left(\vec{\nabla}\phi\right) \times \vec{E} = \frac{\vec{E} \times \vec{\nabla}\phi}{\phi^2} \tag{130}$$

Inserting Eq. (130) into Eq. (127) leads to

$$\vec{\nabla} \cdot \left[\frac{\vec{E} + \vec{\nabla}\phi}{\phi} \times \vec{A}\right] = \frac{\vec{A} \cdot \left(\vec{E} \times \vec{\nabla}\phi\right)}{\phi^2} - \frac{\left(\vec{E} + \vec{\nabla}\phi\right) \cdot \left(\vec{\nabla} \times \vec{A}\right)}{\phi} \tag{131}$$

Inserting Eq. (131) into Eq. (122) results in

$$\vec{A} \cdot \left(\vec{E} \times \vec{\nabla}\phi\right) - \phi\left(\vec{E} + \vec{\nabla}\phi\right) \cdot \left(\vec{\nabla} \times \vec{A}\right) \;=\; 0 \tag{132}$$

Now it is obvious, because Eq. (132) is equivalent to Eq. (122), that the four Eqs. (122) and (123) are indeed first-order differential equations. Compared to the eight second-order differential equations (77) $-$ (80) with eight variables and the five second order differential equations (87) $-$ (89) with five variables, the Eqs. (122) $-$ (124) represent a significant simplification because they are four first order differential equations with four variables. Possibly there is an analytical solution of Eqs. (122) $-$ (124). In contrast to the (nearly) arbitrary scalar potential $\sigma(\vec{r})$ used in section 1, the related scalar potential $g(\vec{r})$ in Eqs. (122) and (123) appears apriori not as an ambiguous quantity. However, the presence of the scalar field $\psi(\vec{r})$ and rotational fields and operators in Eqs. (122) $-$ (124) indicates the existence of an ambiguousness in the overall solution.

We note that there is also another way to derive Eqs. (122) and (123), namely by using Eqs. (81) and (82). Solving Eq. (81) for $\vec{\omega}$ yields

$$\vec{\omega} = \frac{\vec{E} + \vec{\nabla}\phi + \omega_0\,\vec{A}}{\phi} \tag{133}$$

Inserting this into Eq. (82) and by using $\vec{A} \times \vec{A} = 0$ we get

$$-\frac{\vec{E} + \vec{\nabla}\phi}{\phi} \times \vec{A} \;+\; \vec{\nabla} \times \vec{A} \;=\; \vec{B} \tag{134}$$

which is identical to Eq. (123) if $\vec{\nabla}\psi = 0$. By taking the divergence on both sides of Eq. (134) and by means of $\vec{\nabla} \cdot \vec{B} = 0$, which follows from Eq. (126) and means the absence of magnetic monopoles, we obtain Eq. (122).

The Eqs. (122) $-$ (126) imply the following special cases:

- Absence of charge density ρ , i.e. $\rho = 0$ and thus $\vec{E} = 0$

- Absence of current density $\vec{J}$, i.e. $\vec{J} = 0$ and thus $\vec{B} = 0$
- Absence of charge density ρ and current density $\vec{J}$, i.e. $\rho = 0$ and $\vec{J} = 0$ and thus $\vec{E} = \vec{B} = 0$. In this case the solutions of Eqs. (122) $-$ (124) represent possible level II vacuum potentials in the absence of level I fields. This case is considered in section 2.4.

2.4 The equations for the level II potentials in the absence of level I fields: The vacuum equations

Recently H. Eckardt and D. W. Lindstrom have published a paper about the solutions of the latest set of electrodynamic ECE equations in the absence of level I electric and magnetic fields [25]. They point to the existence of non-vanishing vacuum potentials [25].

In the follwing we will study the former set of electro- and magnetostatic ECE equations in the absence of level I electric and magnetic fields, i.e. for $\vec{B} = \vec{E} = 0$. Their solutions indicate the existence of non-vanishing level II vacuum potentials and fields.

The absence of a charge density ρ and current density $\vec{J}$, i.e. $\rho = 0$ and $\vec{J} = 0$, implies $\vec{E} = \vec{B} = 0$ and Eqs. (122) $-$ (124) result in

$$\left[\vec{\nabla} \ln\left(\frac{\phi}{\phi_0}\right)\right] \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0 \tag{135}$$

$$-\left[\vec{\nabla} \ln\left(\frac{\phi}{\phi_0}\right)\right] \times \vec{A} + \vec{\nabla} \times \vec{A} = \vec{\nabla}\psi \tag{136}$$

$$\Delta\,\psi = 0 \tag{137}$$

whereby ϕ_0 is a constant. We recall that the scalar field $\psi(\vec{r})$ represents the presence of an ambiguousness in the solutions. The solutions of Eqs. (135) $-$ (137) represent possible vacuum potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ in the absence of level I fields $\vec{E}$ and $\vec{B}$. We note that in electrodynamics for $\rho = 0$ and $\vec{J} = 0$ another type of vacuum solutions exist, namely such with non-vanishing time-dependent fields $\vec{E}$ and $\vec{B}$. In electrostatics, however, $\rho = 0$ and $\vec{J} = 0$ always implies $\vec{E} = \vec{B} = 0$.

For $\vec{\nabla}\psi = 0$ the Eq. (135) is no longer independent from Eq. (136) because Eq. (136) results in

$$\vec{\nabla} \times \vec{A} = \left[\vec{\nabla} \ln\left(\frac{\phi}{\phi_0}\right)\right] \times \vec{A} \tag{138}$$

Inserting this into Eq. (135) implies a scalar triple product which always vanishes because two of the three involved vectors are equal.

Thus for $\vec{\nabla}\psi = 0$ the level II vacuum potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ in the absence of level I fields are determined only by Eq. (138) which can also be written as

$$\left(\vec{\nabla}\xi\right) \times \vec{A} = \vec{\nabla} \times \vec{A} \tag{139}$$

whereby

$$\xi(\vec{r}) = \ln\left(\frac{\phi(\vec{r})}{\phi_0}\right) \tag{140}$$

We note that if a vector potential $\vec{A}(\vec{r})$ satisfies Eqs. (139) and (140) or Eq. (138), then because of $\vec{\nabla} \times \left(\vec{\nabla}\xi\right) = 0$ and $\left(\vec{\nabla}\xi\right) \times \left(\vec{\nabla}\xi\right) = 0$ also

$$\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + c\,\vec{\nabla}\ln\left(\frac{\phi(\vec{r})}{\phi_0}\right) \tag{141}$$

is a solution of Eqs. (139) and (140) or Eq. (138) whereby c is a constant.

In the following sections 2.4.1 − 2.4.6 we present some solutions of Eqs. (135) − (137), Eq. (138) or Eqs. (139) and (140).

From Eqs. (93) − (96) and (107) − (110) we infer for $\rho = 0$ and $\vec{J} = \vec{E} = \vec{B} = 0$ for the level II potentials and fields the following relations which are associated with Eqs. (135) − (137) and their solutions:

$$\phi(\vec{r}) = g(\vec{r}) \tag{142}$$

$$\vec{E}_\phi(\vec{r}) = \vec{E}_g(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) \tag{143}$$

$$\vec{A}(\vec{r}) = \vec{V}(\vec{r}) \tag{144}$$

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) \tag{145}$$

We recall that these relations mean that all level I potentials and fields vanish, i.e.

$$\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = 0 \tag{146}$$

$$\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r}) = 0 \tag{147}$$

$$\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r}) = 0 \tag{148}$$

$$\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r}) = 0 \tag{149}$$

2.4.1 Solutions with vanishing level II fields

The most simple solution of Eqs. (135) − (137) is given by

$$\phi(\vec{r}) = \phi_1 \tag{150}$$

$$\vec{A}(\vec{r}) = \vec{A}_0 \tag{151}$$

$$\psi(\vec{r}) = \psi_0 \tag{152}$$

whereby ϕ_1 , $\vec{A}_0$ and ψ_0 are constants. According to Eqs. (143) and (145) the associated level II fields vanish, i.e.

$$\vec{E}_\phi(\vec{r}) = \vec{E}_g(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) = 0 \tag{153}$$

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = 0 \tag{154}$$

This case corresponds to the vacuum solutions of textbook electro- and magnetostatics.

2.4.2 Solutions with vanishing level II magnetic field

Another solutions are those which imply

$$\psi(\vec{r}) = \psi_0 \tag{155}$$

and a curl-free vector potential, i.e.

$$\vec{\nabla} \times \vec{A}(\vec{r}) = 0 \tag{156}$$

whereby ψ_0 is a constant. In this case, according to Eq. (145), the associated level II magnetic fields vanish:

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = 0 \tag{157}$$

From Eq. (156) we infer

$$\vec{A}(\vec{r}) = \vec{\nabla}\xi(\vec{r}) \tag{158}$$

whereby $\xi(\vec{r})$ is any scalar field. In this case the cross product in Eq. (136) has to be zero and thus the two vectors $\vec{\nabla}\ \ln(\phi/\phi_0)$ and $\vec{A}$ are (anti)parallel to each other, i.e.

$$\vec{\nabla}\ \ln\left(\frac{\phi}{\phi_0}\right) = c\,\vec{A} = c\,\vec{\nabla}\xi(\vec{r}) \tag{159}$$

and thus

$$\phi(\vec{r}) = \phi_0\ \exp\Big(c\,\xi(\vec{r})\Big) \tag{160}$$

whereby c is a constant. Inserting Eq. (160) into Eq. (143) yields for the associated level II electric fields

$$\vec{E}_\phi(\vec{r}) = \vec{E}_g(\vec{r}) = -\,\vec{\nabla}\phi(\vec{r}) = -\,c\,\phi_0\,\exp\Big(c\,\xi(\vec{r})\Big)\,\vec{\nabla}\xi(\vec{r}) \tag{161}$$

2.4.3 Solutions with constant level II magnetic field

Further solutions are those which imply the most simple case of $\vec{\nabla}\times\vec{A}(\vec{r}) \neq 0$, namely

$$\vec{\nabla}\times\vec{A}(\vec{r}) = \vec{B}_{A\,0} \tag{162}$$

whereby $\vec{B}_{A\,0}$ is a constant vector. In this case, according to Eq. (145), the associated level II magnetic fields are

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \vec{B}_{A\,0} \tag{163}$$

A simple example of a vector potential $\vec{A}(\vec{r})$ which satisfies Eq. (162) is given by [19]

$$\vec{A}(\vec{r}) = \frac{B_{A\,0}}{2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \tag{173}$$

which yields

$$\vec{\nabla} \times \vec{A}(\vec{r}) = B_{A\,0} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{174}$$

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[19] A somewhat more complicated vector potential $\vec{A}(\vec{r})$ which implies a constant curl results from the ansatz

$$\vec{A}(\vec{r}) = f(r) \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \tag{164}$$

which leads to

$$\vec{\nabla} \times \vec{A} = \left[2\, f(r) \;+\; r\, \frac{d\, f(r)}{d\, r} \right] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{165}$$

whereby $f(r)$ is any function of

$$r = \sqrt{x^2 + y^2} \tag{166}$$

Eq. (165) represents a constant vector if

$$2\, f(r) + r\, \frac{d\, f}{d\, r} = B_{A\,0} \tag{167}$$

whereby $B_{A\,0}$ is a constant. Eq. (167) corresponds to a differential equation of the type

$$\frac{d\, f}{d\, r} + u(r)\, f(r) = v(r) \tag{168}$$

with

$$u(r) = \frac{2}{r} \quad \text{and} \quad v(r) = \frac{B_{A\,0}}{r} \tag{169}$$

The general solution of Eq. (168) is well-known, namely

$$f(r) = \exp\!\left(-\int u(r)\, dr \right) \left[f_0 + \int v(r)\, \exp\!\left(\int u(r)\, dr \right) dr \right] \tag{170}$$

whereby f_0 is a constant. Inserting Eqs. (169) into Eq. (170) results in

$$f(r) = \frac{B_{A\,0}}{2} + \frac{f_0}{r^{\,2}} \tag{171}$$

Inserting Eqs. (171) and (166) into Eqs. (164) and (165) yields

$$\vec{A}(\vec{r}) = \left[\frac{f_0}{x^{\,2} + y^{\,2}} + \frac{B_{A\,0}}{2} \right] \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad \Rightarrow \quad \vec{\nabla} \times \vec{A} = B_{A\,0} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{172}$$

whereby f_0 and $B_{A\,0}$ are two parameters. Thus the curl of this vector potential $\vec{A}(\vec{r})$ is a constant vector which depends only on $B_{A\,0}$ but not on f_0.

whereby $B_{A0} = |\, \vec{B}_{A0} \,|$ is a constant. Now let's try to find a scalar potential $\phi(\vec{r})$ which solves Eqs. (135) – (137) when the vector potential $\vec{A}(\vec{r})$ is given by Eq. (173). Inserting Eq. (174) into Eq. (135) yields

$$\left[\vec{\nabla} \ln\left(\frac{\phi}{\phi_0}\right)\right] \cdot \left(\vec{\nabla} \times \vec{A}\right) = B_{A0} \frac{\partial}{\partial z} \ln\left(\frac{\phi}{\phi_0}\right) = 0 \tag{175}$$

For $B_{A0} \neq 0$ this implies

$$\frac{\partial}{\partial z} \ln\left(\frac{\phi}{\phi_0}\right) = 0 \tag{176}$$

and thus $\phi(\vec{r})$ does not depend on z , i.e.

$$\phi = \phi(x, y) \tag{177}$$

Inserting Eqs. (173) and (174) into Eq. (136) and taking into account Eq. (177) leads to

$$0 = \frac{\partial \psi}{\partial x} \tag{178}$$

$$0 = \frac{\partial \psi}{\partial y} \tag{179}$$

$$-\frac{B_{A0}}{2}\left[x \frac{\partial}{\partial x} \ln\left(\frac{\phi}{\phi_0}\right) + y \frac{\partial}{\partial y} \ln\left(\frac{\phi}{\phi_0}\right)\right] + B_{A0} = \frac{\partial \psi}{\partial z} \tag{180}$$

According to Eq. (177) the scalar potential ϕ does not depend on z and thus we infer from Eqs. (178) – (180)

$$\psi = \psi_0 + b\, z \tag{181}$$

whereby ψ_0 and b are constants. Thus Eq. (180) results in

$$x \frac{\partial}{\partial x} \ln\left(\frac{\phi}{\phi_0}\right) + y \frac{\partial}{\partial y} \ln\left(\frac{\phi}{\phi_0}\right) = \frac{2\,(B_{A0} - b\,)}{B_{A0}} \tag{182}$$

A solution of this partial differential equation is given by

$$\ln\left(\frac{\phi}{\phi_0}\right) = \frac{2\,(B_{A0} - b\,)}{B_{A0}} \ln\left(\frac{\sqrt{x^2 + y^2}}{r_0}\right) \tag{183}$$

whereby r_0 is a constant, and thus

$$\begin{aligned} \phi(\vec{r}) &= \phi_0 \exp\left[\frac{2\,(B_{A0} - b\,)}{B_{A0}} \ln\left(\frac{\sqrt{x^2 + y^2}}{r_0}\right)\right] \\ &= \phi_0 \left(\frac{\sqrt{x^2 + y^2}}{r_0}\right)^{\dfrac{2\,B_{A0} \,-\, 2\,b}{B_{A0}}} \end{aligned} \tag{184}$$

Inserting Eq. (184) into Eq. (143) yields the associated level II electric fields, namely

$$\vec{E}_\phi(\vec{r}) = \vec{E}_g(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) = -\phi_0 \left(\frac{\sqrt{x^2+y^2}}{r_0}\right)^{-\frac{2b}{B_{A0}}} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \tag{185}$$

2.4.4 Solutions with level II magnetic and electric fields which both depend on the position vector

Another solutions are those which imply a vector potential $\vec{A}(\vec{r})$ whose curl is not constant. A relatively simple example of such a vector potential is

$$\vec{A}(\vec{r}) = a_0 \left(x^2+y^2\right)^n \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \tag{186}$$

and thus

$$\vec{\nabla} \times \vec{A}(\vec{r}) = 2\, a_0\, (n+1) \left(x^2+y^2\right)^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{187}$$

whereby a_0 and n are constants. In this case, according to Eq. (145), the associated level II magnetic fields are

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = 2\, a_0\, (n+1) \left(x^2+y^2\right)^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{188}$$

Now let's try to find a scalar potential $\phi(\vec{r})$ which solves Eqs. (135) $-$ (137) when the vector potential $\vec{A}(\vec{r})$ is given by Eq. (186). Inserting Eq. (187) into Eq. (135) yields

$$\left[\vec{\nabla} \ln\left(\frac{\phi}{\phi_0}\right)\right] \cdot \left(\vec{\nabla} \times \vec{A}\right) = 2\, a_0\, (n+1) \left(x^2+y^2\right)^n \frac{\partial}{\partial z} \ln\left(\frac{\phi}{\phi_0}\right) = 0 \tag{189}$$

For

$$a_0\, (n+1) \left(x^2+y^2\right)^n \neq 0 \tag{190}$$

Eq. (189) implies

$$\frac{\partial}{\partial z} \ln\left(\frac{\phi}{\phi_0}\right) = 0 \tag{191}$$

and thus $\phi(\vec{r})$ does not depend on z , i.e.

$$\phi = \phi(x, y) \tag{192}$$

Inserting Eqs. (186) and (187) into Eq. (136) and taking into account Eq. (192) leads to

$$\frac{\partial\,\psi}{\partial\,x} = 0 \tag{193}$$

$$\frac{\partial\,\psi}{\partial\,y} = 0 \tag{194}$$

$$\begin{aligned}\frac{\partial\,\psi}{\partial\,z} = &- \; a_0 \left(x^{\,2} + y^{\,2}\right)^n \left[x\,\frac{\partial}{\partial\,x}\ln\!\left(\frac{\phi}{\phi_0}\right) + y\,\frac{\partial}{\partial\,y}\ln\!\left(\frac{\phi}{\phi_0}\right)\right] \\ &+ \; 2\,a_0\,(n+1)\left(x^{\,2} + y^{\,2}\right)^n\end{aligned} \tag{195}$$

For the sake of simplicity we choose

$$\psi = \psi_0 \tag{196}$$

whereby ψ_0 is a constant. In this case Eq. (195) results in

$$x\,\frac{\partial}{\partial\,x}\ln\!\left(\frac{\phi}{\phi_0}\right) + y\,\frac{\partial}{\partial\,y}\ln\!\left(\frac{\phi}{\phi_0}\right) = 2\,(n+1) \tag{197}$$

A solution of this partial differential equation is given by

$$\ln\!\left(\frac{\phi}{\phi_0}\right) = 2\,(n+1)\,\ln\!\left(\frac{\sqrt{x^{\,2} + y^{\,2}}}{r_0}\right) \tag{198}$$

whereby r_0 is a constant, and thus

$$\begin{aligned}\phi(\vec{r}) &= \phi_0\,\exp\!\left[2\,(n+1)\,\ln\!\left(\frac{\sqrt{x^{\,2} + y^{\,2}}}{r_0}\right)\right] \\ &= \phi_0\left(\frac{\sqrt{x^{\,2} + y^{\,2}}}{r_0}\right)^{2\,(n+1)}\end{aligned} \tag{199}$$

Inserting Eq. (199) into Eq. (143) yields the associated level II electric fields, namely

$$\vec{E}_\phi\,(\vec{r}) = \vec{E}_g\,(\vec{r}) = -\,\vec{\nabla}\,\phi(\vec{r}) = -\,\phi_0\left(\frac{\sqrt{x^{\,2} + y^{\,2}}}{r_0}\right)^{2\,n}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \tag{200}$$

2.4.5 A class of more general solutions

Further solutions of Eq. (138) or (139) are the following. By inserting the ansatz

$$\vec{A}(\vec{r}) = a(\vec{r}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{201}$$

into Eq. (139) we obtain

$$\frac{\partial a}{\partial y} - \frac{\partial a}{\partial z} = a \frac{\partial \xi}{\partial y} - a \frac{\partial \xi}{\partial z} \tag{202}$$

$$\frac{\partial a}{\partial z} - \frac{\partial a}{\partial x} = a \frac{\partial \xi}{\partial z} - a \frac{\partial \xi}{\partial x} \tag{203}$$

$$\frac{\partial a}{\partial x} - \frac{\partial a}{\partial y} = a \frac{\partial \xi}{\partial x} - a \frac{\partial \xi}{\partial y} \tag{204}$$

whereby $a(\vec{r})$ is any scalar function. These equations are satisfied if

$$\xi(\vec{r}) = \ln \left(\frac{a(\vec{r})}{a_0} \right) \tag{205}$$

whereby a_0 is a constant. Because of Eq. (140) this implies

$$\frac{a(\vec{r})}{a_0} = \frac{\phi(\vec{r})}{\phi_0} \tag{206}$$

and thus Eq. (201) becomes

$$\vec{A}(\vec{r}) = \phi(\vec{r}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{207}$$

whereby $\phi(\vec{r})$ is any scalar potential [20]. The vector potentials given by Eq. (207) represent a general class of solutions of Eq. (138). By taking into account Eq. (141) the most general type of these solutions is given by

$$\vec{A}(\vec{r}) \; = \; \phi(\vec{r}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c \, \vec{\nabla} \ln \left(\frac{\phi(\vec{r})}{\phi_0} \right) \tag{208}$$

whereby $\phi(\vec{r})$ is any scalar potential and ϕ_0 and c are constants. Their associated level II electric and magnetic fields are given by Eqs. (143) and (145), namely

$$\vec{E}_\phi(\vec{r}) = \vec{E}_g(\vec{r}) = - \vec{\nabla} \phi(\vec{r}) \tag{209}$$

[20] Also related vector potentials like $\vec{A}(\vec{r}) = \phi(\vec{r}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\vec{A}(\vec{r}) = \phi(\vec{r}) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ are solutions of Eq. (138)

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = \begin{pmatrix} \dfrac{\partial \phi}{\partial y} - \dfrac{\partial \phi}{\partial z} \\ \\ \dfrac{\partial \phi}{\partial z} - \dfrac{\partial \phi}{\partial x} \\ \\ \dfrac{\partial \phi}{\partial x} - \dfrac{\partial \phi}{\partial y} \end{pmatrix} \tag{210}$$

2.4.6 Another class of more general solutions

Another solutions of Eq. (138) or (139) are the following. By inserting the ansatz

$$\vec{A}(\vec{r}) = A_0 \begin{pmatrix} b(y)\, c(z) \\ \\ a(x)\, c(z) \\ \\ a(x)\, b(y) \end{pmatrix} \tag{211}$$

into Eq. (139) we obtain

$$\frac{d\,b}{d\,y} - \frac{d\,c}{d\,z} = b(y)\,\frac{\partial\,\xi}{\partial\,y} - c(z)\,\frac{\partial\,\xi}{\partial\,z} \tag{212}$$

$$\frac{d\,c}{d\,z} - \frac{d\,a}{d\,x} = c(z)\,\frac{\partial\,\xi}{\partial\,z} - a(x)\,\frac{\partial\,\xi}{\partial\,x} \tag{213}$$

$$\frac{d\,a}{d\,x} - \frac{d\,b}{d\,y} = a(x)\,\frac{\partial\,\xi}{\partial\,x} - b(y)\,\frac{\partial\,\xi}{\partial\,y} \tag{214}$$

whereby A_0 is a constant and $a(x)$, $b(y)$ and $c(z)$ are any functions which depend only on x, y and z, repectively. These equations are satisfied if

$$\xi(\vec{r}) = \ln\left(\, a(x)\, b(y)\, c(z)\, \right) \tag{215}$$

Because of Eq. (140) this implies

$$a(x)\, b(y)\, c(z) = \frac{\phi(\vec{r})}{\phi_0} \tag{216}$$

and thus the scalar potential $\phi(\vec{r})$ is given by

$$\phi(\vec{r}) = \phi_0\, a(x)\, b(y)\, c(z) \tag{217}$$

whereby ϕ_0 is a constant. The vector potentials and scalar potentials which are given by Eqs. (211) and (217) represent a general class of solutions of Eq. (138). By taking into account Eq. (141) the most general type of these solutions is given by

$$\vec{A}(\vec{r}) \;=\; A_0 \begin{pmatrix} b(y)\,c(z) \\ \\ a(x)\,c(z) \\ \\ a(x)\,b(y) \end{pmatrix} \;+\; K_0 \begin{pmatrix} \dfrac{1}{a(x)}\,\dfrac{d\,a}{d\,x} \\ \\ \dfrac{1}{b(y)}\,\dfrac{d\,b}{d\,y} \\ \\ \dfrac{1}{c(z)}\,\dfrac{d\,c}{d\,z} \end{pmatrix} \tag{218}$$

$$\phi(\vec{r}) = \phi_0\, a(x)\, b(y)\, c(z) \tag{219}$$

whereby A_0, K_0 and ϕ_0 are constants and $a(x)$, $b(y)$ and $c(z)$ are any functions which depend only on x, y and z, repectively. Their associated level II electric and magnetic fields are given by Eqs. (143) and (145), namely

$$\vec{E}_\phi(\vec{r}) = \vec{E}_g(\vec{r}) = -\,\vec{\nabla}\phi(\vec{r}) = -\,\phi_0 \begin{pmatrix} \dfrac{d\,a}{d\,x}\, b(y)\, c(z) \\ \\ a(x)\,\dfrac{d\,b}{d\,y}\, c(z) \\ \\ a(x)\, b(y)\,\dfrac{d\,c}{d\,z} \end{pmatrix} \tag{220}$$

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = A_0 \begin{pmatrix} a(x)\,\dfrac{d\,b}{d\,y} - a(x)\,\dfrac{d\,c}{d\,z} \\ \\ b(y)\,\dfrac{d\,c}{d\,z} - b(y)\,\dfrac{d\,a}{d\,x} \\ \\ c(z)\,\dfrac{d\,a}{d\,x} - c(z)\,\dfrac{d\,b}{d\,y} \end{pmatrix} \tag{221}$$

2.4.7 The meaning of the vacuum solutions

The solutions presented in the previous sections 2.4.1 $-$ 2.4.6 represent possible level II vacuum potentials and fields. We recall that according to Eqs. (142) $-$ (149) the levell II vacuum potentials and fields sum up to zero [21] at every location so that level I potentials and fields do not appear. The solutions presented in the sections 2.4.1 $-$ 2.4.6 represent an infinite number of different electromagnetic vacuum potentials and fields. Within the framework of electro- and magnetostatics there are no obvious (boundary) conditions which specify a concrete type. Therefore the solutions presented in the sections 2.4.1 $-$ 2.4.6 mean that electromagnetic vacuum potentials and fields are possible or exist, even if their concrete form remains an open question. Concerning this issue the following should be noted:

[21] The feature that the level I potentials and fields emerge from a difference of two level II potentials or fields, see e.g. Eqs. (93), (94), (107), (108) and (146) $-$ (149), can also be described as a sum of two quantities, for example $\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \phi(\vec{r}) + (\,-\,g(\vec{r})\,)$.

- The consideration of electromagnetic vacuum states within the framework of electro- and magnetostatics represents a rough approach and electrodynamics is certainly more appropriate to address this issue.
- The actual vacuum states are not only determined by electromagnetic potentials and fields but also by other contributions such as gravitational potentials and fields, and their mutual interaction.
- Even if the vacuum constitutes the overwhelming part of the universe, matter like electrically charged particles is also present. Therefore it seems likely that the actual vacuum potentials and fields are influenced by the presence of matter.

2.4.8 Hypothetical vacuum charge and current densities

According to Eqs. (142) – (149) the level II vacuum potentials and fields sum up to zero [22] at every location so that level I potentials and fields do not appear [23]. Possibly, the presence of level II vacuum potentials and fields implies the existence of level II or vacuum charge and current densities, similar to the level I potentials and fields β , $\vec{A}$, $\vec{E}$ and $\vec{B}$ which are generated by the (level I) charge density ρ and current densitiy $\vec{J}$. The Eqs. (142) – (149) suggest for the hypothetical vacuum charge densities, ρ_ϕ and ρ_g , and hypothetical vacuum current densities, $\vec{J}_A$ and $\vec{J}_V$, the relations [24]

$$\rho_g(\vec{r}) = -\rho_\phi(\vec{r}) \tag{222}$$

$$\vec{J}_V(\vec{r}) = -\vec{J}_A(\vec{r}) \tag{223}$$

so that the total (level I) charge density $\rho(\vec{r})$ and current density $\vec{J}(\vec{r})$ vanishes:

$$\rho(\vec{r}) = \rho_\phi(\vec{r}) + \rho_g(\vec{r}) = 0 \tag{224}$$

$$\vec{J}(\vec{r}) = \vec{J}_A(\vec{r}) + \vec{J}_V(\vec{r}) = 0 \tag{225}$$

It appears presently not clear how to compute the hypothetical vacuum charge and current density, $\rho_\phi(\vec{r})$ and $\vec{J}_A(\vec{r})$, from the vacuum potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$. One possibility is to assume that the relation between the hypothetical vacuum charge

[22] The feature that the level I potentials and fields emerge from a difference of two level II potentials or fields, see e.g. Eqs. (93), (94), (107), (108) and (146) – (149), can also be described as a sum of two quantities, for example $\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \phi(\vec{r}) + (-g(\vec{r}))$.

[23] The presence of (electrically charged) matter, i.e. charge density $\rho \neq 0$ and/or current density $\vec{J} \neq 0$, breaks this symmetry and level I potentials and fields emerge.

[24] The feature that the level I potentials and fields emerge from a difference of two level II potentials or fields, see e.g. Eqs. (93), (94), (107), (108) and (146) – (149), can also be described as a sum of two quantities, for example $\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \phi(\vec{r}) + (-g(\vec{r}))$.

and current density and the vacuum potentials is of the type given by Eqs. (89) and (106). However, the decoupled linear seconder-order differential equations (89) and (106) describe the behavior of level I quantities, whereas the vacuum potentials are level II quantities which are specified by the coupled non-linear first-order differential equations (135) – (137) or (139) – (140). Thus the relationship between the vacuum potentials and the hypothetical vacuum charge and current density remains an open question.

Furthermore, the consideration of electromagnetic vacuum states within the framework of electro- and magnetostatics represents a rough approach and electrodynamics is certainly more appropriate to address this issue.

2.5 The equations for the level II potentials in the absence of level I magnetic fields

In the following section we present level II potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ which solve Eqs. (122) – (124) when there is no current density, i.e. $\vec{J}(\vec{r}) = 0$, and thus no level I magnetic field, i.e. $\vec{B}(\vec{r}) = 0$.
In Eqs. (122) – (124) we choose for the sake of simplicity $\vec{\nabla}\psi = 0$. Then for $\vec{B} = 0$ Eqs. (122) and (123) result in

$$\vec{\nabla} \cdot \left[\frac{\vec{E} + \vec{\nabla}\phi}{\phi} \times \vec{A} \right] = 0 \tag{226}$$

$$\frac{\vec{E} + \vec{\nabla}\phi}{\phi} \times \vec{A} = \vec{\nabla} \times \vec{A} \tag{227}$$

Inserting Eq. (227) into Eq. (226) yields

$$\vec{\nabla} \cdot \left[\vec{\nabla} \times \vec{A} \right] = 0 \tag{228}$$

which is valid for any $\vec{A}(\vec{r})$. Thus, if there is a scalar potential $\phi(\vec{r})$ and a vector potential $\vec{A}(\vec{r})$ which solves Eq. (227), then also Eq. (226) is fulfilled.

2.5.1 Presentation of a solution for any charge density

Now let's try to find a solution of Eq. (227) by the assumption or ansatz that the first term on the left-hand side of Eq. (227) can be represented as a gradient of a scalar field $\xi(\vec{r})$, i.e.

$$\frac{\vec{E} + \vec{\nabla}\phi}{\phi} = \vec{\nabla}\xi(\vec{r}) \tag{229}$$

In this case Eq. (227) is of the same type as Eq. (139). Solutions of Eq. (139) are presented in sections 2.4.1 − 2.4.6. For example, from section 2.4.5 we know that

$$\vec{A}(\vec{r}) = \kappa(\vec{r}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{230}$$

is a solution of

$$\left(\vec{\nabla}\xi\right) \times \vec{A} = \vec{\nabla} \times \vec{A} \tag{231}$$

when

$$\xi(\vec{r}) = \ln\left(\kappa(\vec{r})\right) \tag{232}$$

whereby $\kappa(\vec{r})$ is any scalar function. Inserting Eq. (232) into Eq. (229) leads to

$$\vec{E} + \vec{\nabla}\phi \;=\; \phi\, \vec{\nabla}\, \ln \kappa \tag{233}$$

We recall that $\vec{E} = -\, \vec{\nabla}\beta$ is considered as a known function which is given by the charge density ρ , see Eqs. (94) and (93). By inserting $\vec{E} = -\, \vec{\nabla}\beta$ into Eq. (233) we obtain

$$-\, \vec{\nabla}\beta + \vec{\nabla}\phi \;=\; \phi\, \vec{\nabla}\, \ln \kappa \tag{234}$$

Now we try to find a level II scalar potential ϕ that solves Eq. (234) which comprises the three following equations:

$$\frac{\partial\, \phi}{\partial\, x} - \phi\, \frac{\partial}{\partial\, x} \ln \kappa \;=\; \frac{\partial\, \beta}{\partial\, x} \tag{235}$$

$$\frac{\partial\, \phi}{\partial\, y} - \phi\, \frac{\partial}{\partial\, y} \ln \kappa \;=\; \frac{\partial\, \beta}{\partial\, y} \tag{236}$$

$$\frac{\partial\, \phi}{\partial\, z} - \phi\, \frac{\partial}{\partial\, z} \ln \kappa \;=\; \frac{\partial\, \beta}{\partial\, z} \tag{237}$$

ϕ is specified by three equations. This means that ϕ is over-determined and therefore it could be that no solution exists. However, in the case of Eqs. (235) − (237) we will now show that there are solutions ϕ when κ is chosen appropriately,

The solution ϕ of the separate Eqs. (235), (236) and (237) are known, see Eqs. (168) and (170) in footnote 19 on page 34, namely

$$\phi = \exp\left[-\int \left(-\frac{\partial}{\partial\, x} \ln \kappa\right) dx\right] \left\{\phi_0 + \int \frac{\partial\, \beta}{\partial\, x}\, \exp\left[\int \left(-\frac{\partial}{\partial\, x} \ln \kappa\right) dx\right] dx\right\} \tag{238}$$

$$\phi = \exp\left[-\int \left(-\frac{\partial}{\partial\, y} \ln \kappa\right) dy\right] \left\{\phi_0 + \int \frac{\partial\, \beta}{\partial\, y}\, \exp\left[\int \left(-\frac{\partial}{\partial\, y} \ln \kappa\right) dy\right] dy\right\} \tag{239}$$

$$\phi = \exp\left[-\int\left(-\frac{\partial}{\partial z}\ln\kappa\right)dz\right]\left\{\phi_0 + \int\frac{\partial\beta}{\partial z}\exp\left[\int\left(-\frac{\partial}{\partial z}\ln\kappa\right)dz\right]dz\right\} \tag{240}$$

whereby ϕ_0 is a constant. These three equations result in

$$\phi = \kappa\left(\phi_0 + \int\frac{1}{\kappa}\frac{\partial\beta}{\partial x}\,dx\right) \tag{241}$$

$$\phi = \kappa\left(\phi_0 + \int\frac{1}{\kappa}\frac{\partial\beta}{\partial y}\,dy\right) \tag{242}$$

$$\phi = \kappa\left(\phi_0 + \int\frac{1}{\kappa}\frac{\partial\beta}{\partial z}\,dz\right) \tag{243}$$

If we choose

$$\kappa(\vec{r}) = \frac{\beta(\vec{r})}{\beta_0} \tag{244}$$

whereby β_0 is a constant, then Eqs. (241) $-$ (243) become

$$\phi = \frac{\phi_0}{\beta_0}\beta + \beta\int\frac{\partial}{\partial x}\ln\left(\frac{\beta}{\beta_0}\right)dx = \frac{\phi_0}{\beta_0}\beta + \beta\,\ln\left(\frac{\beta}{\beta_0}\right) \tag{245}$$

$$\phi = \frac{\phi_0}{\beta_0}\beta + \beta\int\frac{\partial}{\partial y}\ln\left(\frac{\beta}{\beta_0}\right)dy = \frac{\phi_0}{\beta_0}\beta + \beta\,\ln\left(\frac{\beta}{\beta_0}\right) \tag{246}$$

$$\phi = \frac{\phi_0}{\beta_0}\beta + \beta\int\frac{\partial}{\partial z}\ln\left(\frac{\beta}{\beta_0}\right)dz = \frac{\phi_0}{\beta_0}\beta + \beta\,\ln\left(\frac{\beta}{\beta_0}\right) \tag{247}$$

They represent a solution for ϕ because the three different expressions lead finally to the same result, namely

$$\phi = \left[\frac{\phi_0}{\beta_0} + \ln\left(\frac{\beta}{\beta_0}\right)\right]\beta \tag{248}$$

By inserting Eq. (248) into Eq. (93) and by means of Eqs. (95) and (96) we get the level II quantities g , $\vec{E}_\phi$ and $\vec{E}_g$, namely

$$g = \phi - \beta = \left[\frac{\phi_0}{\beta_0} - 1 + \ln\left(\frac{\beta}{\beta_0}\right)\right]\beta \tag{249}$$

$$\vec{E}_\phi = -\,\vec{\nabla}\phi = -\left[\frac{\phi_0}{\beta_0} + 1 + \ln\left(\frac{\beta}{\beta_0}\right)\right]\vec{\nabla}\beta \tag{250}$$

$$\vec{E}_g = -\,\vec{\nabla} g = -\left[\frac{\phi_0}{\beta_0} + \ln\left(\frac{\beta}{\beta_0}\right)\right]\vec{\nabla}\beta \tag{251}$$

Now let's compute the level II magnetic quantities which result from the level II vector potential $\vec{A}$ which is given by Eq. (230). Inserting Eq. (244) into Eq. (230) yields

$$\vec{A} = \frac{\beta}{\beta_0} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{252}$$

By taking into account Eqs. (227) – (232), (244) and (138) – (141) the expression of the level II vector potential $\vec{A}$ which solves Eqs. (227) and (229) can be extended to

$$\vec{A} \;=\; \frac{\beta}{\beta_0} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a_0 \vec{\nabla} \ln\left(\frac{\beta}{\beta_0}\right) \tag{253}$$

whereby a_0 is a constant.
The level II magnetic fields $\vec{B}_A$ and $\vec{B}_V$ can be calculated by our assumption $\vec{B} = 0$ and Eqs. (108) – (110) and (253), namely

$$\vec{B}_A = \vec{B}_V = \vec{\nabla} \times \vec{A} = \frac{1}{\beta_0} \begin{pmatrix} \frac{\partial \beta}{\partial y} - \frac{\partial \beta}{\partial z} \\ \\ \frac{\partial \beta}{\partial z} - \frac{\partial \beta}{\partial x} \\ \\ \frac{\partial \beta}{\partial x} - \frac{\partial \beta}{\partial y} \end{pmatrix} \tag{254}$$

We summarize the results of this section:

We have considered a charge density $\rho(\vec{r}) \neq 0$ in the absence of a current density $\vec{J}(\vec{r})$, i.e. $\vec{J}(\vec{r}) = 0$. According to Eqs. (93) − (96) and (107) − (110) the resulting level I quantities, namely the scalar potential $\beta(\vec{r})$, vector potential $\vec{\Lambda}(\vec{r})$, electric field $\vec{E}(\vec{r})$ and magnetic field $\vec{B}(\vec{r})$ are

$$\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \tag{255}$$

$$\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r}) = -\,\vec{\nabla}\left[\phi(\vec{r}) - g(\vec{r})\right] = -\,\vec{\nabla}\beta(\vec{r}) \tag{256}$$

$$\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r}) = 0 \tag{257}$$

$$\vec{B}(\vec{r}) \;=\; \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r}) \;=\; \vec{\nabla} \times \left[\vec{A}(\vec{r}) - \vec{V}(\vec{r})\right] \;=\; \vec{\nabla} \times \vec{\Lambda}(\vec{r}) = 0 \tag{258}$$

For Eq. (227), which determines the level II potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$, a solution was found by means of an ansatz or assumption which is given by Eq. (229). The corresponding level II quantities, namely the scalar potential $\phi(\vec{r})$, electric field $\vec{E}_\phi(\vec{r})$, vector potential $\vec{A}(\vec{r})$ and magnetic field $\vec{B}_A(\vec{r})$, are

$$\phi(\vec{r}) = \left[\frac{\phi_0}{\beta_0} + \ln\left(\frac{\beta(\vec{r})}{\beta_0}\right)\right]\beta(\vec{r}) \tag{259}$$

$$\vec{E}_\phi(\vec{r}) = -\,\vec{\nabla}\phi(\vec{r}) = -\left[\frac{\phi_0}{\beta_0} + 1 + \ln\left(\frac{\beta(\vec{r})}{\beta_0}\right)\right]\vec{\nabla}\beta(\vec{r}) \tag{260}$$

$$\vec{A}(\vec{r}) = \frac{\beta(\vec{r})}{\beta_0}\begin{pmatrix}1\\1\\1\end{pmatrix} + a_0\,\vec{\nabla}\ln\left(\frac{\beta(\vec{r})}{\beta_0}\right) \tag{261}$$

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = \frac{1}{\beta_0}\begin{pmatrix} \dfrac{\partial\beta(\vec{r})}{\partial\, y} - \dfrac{\partial\beta(\vec{r})}{\partial\, z} \\[2ex] \dfrac{\partial\beta(\vec{r})}{\partial\, z} - \dfrac{\partial\beta(\vec{r})}{\partial\, x} \\[2ex] \dfrac{\partial\beta(\vec{r})}{\partial\, x} - \dfrac{\partial\beta(\vec{r})}{\partial\, y} \end{pmatrix} \tag{262}$$

whereby ϕ_0 , β_0 and a_0 are constants.

The results indicate that an electric charge density or an electric charge is accompanied by non-vanishing level II magnetic quantities like the vector potential $\vec{A}(\vec{r})$ and magnetic field $\vec{B}_A(\vec{r})$, even if a current density $\vec{J}(\vec{r})$ and a level I magnetic field $\vec{B}(\vec{r})$ are absent, i.e. $\vec{J}(\vec{r}) = \vec{B}(\vec{r}) = 0$. We note that the level II potentials which are described by Eqs. (259) and (261), and their associated level II fields, represent just one of many possible solutions of Eq. (227). It seems that the ECE electro- and magnetostatics does not provide constraints which favors one specific solution for the level II quantities.

2.5.2 A further type of level II potentials which might appear as another solution and why they do not represent a genuine solution

coming soon ...

2.6 Summary

By the introduction of two further quantities, namely the scalar potential $g(\vec{r})$ and vector potential $\vec{V}(\vec{r})$, the original electro- and magnetostatic ECE equations (77) – (82) were transformed into a set of equations which allow a direct comparison with the equations of textbook electro- and magnetostatics. The transformed ECE equations reveal that ECE electro- and magnetostatics represents an extension of textbook electro- and magnetostatics.
They are compatible with each other in the sense that textbook electro- and magnetostatics represents a special case of ECE electro- and magnetostatics.

On the level of the quantities

$$\text{scalar potential } \beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) \tag{263}$$

$$\text{electric field } \vec{E}(\vec{r}) = -\vec{\nabla}\beta(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r}) \tag{264}$$

$$\text{vector potential } \vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r}) \tag{265}$$

$$\text{magnetic field } \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{\Lambda}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r}) \tag{266}$$

textbook and ECE electro- and magnetostatics are equivalent. This means that the scalar potential $\beta(\vec{r})$, electric field $\vec{E}(\vec{r})$, vector potential $\vec{\Lambda}(\vec{r})$ and magnetic field $\vec{B}(\vec{r})$ is identical with the scalar potential, electric field, vector potential and magnetic field of textbook electro- and magnetostatics, respectively. In ECE electro- and magnetostatics these so-called level I potentials and fields appear as difference between two so-called level II quantitites which both depend on $\vec{r}$, whereas in textbook electro- and magnetostatics they result from always one spatially dependent function which corresponds to the case $\vec{\nabla} g = 0$ and $\vec{\nabla} \times \vec{V} = 0$, i.e. $g(\vec{r}) = g_0$ and $\vec{V} = \vec{\nabla}\alpha(\vec{r})$ whereby g_0 is a constant and $\alpha(\vec{r})$ any scalar field.

The transformed ECE equations decompose into two sets of equations and quantities. The first set corresponds to the equations of textbook electro- and magnetostatics which determine the level I potentials and fields. The second set of equations specify the level II potentials and fields.

The level I potentials $\beta(\vec{r})$ and $\vec{\Lambda}(\vec{r})$ are determined by the four well-known linear decoupled second-order differential equations of textbook electro- and magnetostatics, namely

$$\Delta\,\beta = -\frac{\rho}{\epsilon_0} \tag{267}$$

$$\Delta\,\vec{\Lambda} = -\,\mu_0\,\vec{J} \tag{268}$$

whereby $\rho(\vec{r})$ is the charge density and $\vec{J}(\vec{r})$ the current density. For spatially limited charge and current densities their solution is well-known, namely [25] [26]

$$\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\;d^3R \tag{269}$$

$$\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r}) = \frac{\mu_0}{4\,\pi}\iiint \frac{\vec{J}(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\;d^3R \tag{270}$$

The level II potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ are determined by the following four non-linear coupled first-order differential equations

$$\vec{\nabla}\cdot\left[\frac{\vec{E} + \vec{\nabla}\phi}{\phi}\times\vec{A}\right] = \;0 \tag{271}$$

$$-\frac{\vec{E} + \vec{\nabla}\phi}{\phi}\times\vec{A}\;+\;\vec{\nabla}\times\vec{A}\;=\;\vec{B} \tag{272}$$

whereby the fields

$$\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r}) = -\,\vec{\nabla}\beta(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{\rho(\vec{R})\,(\vec{r} - \vec{R})}{|\,\vec{r} - \vec{R}\,|^3}\;d^3R \tag{273}$$

$$\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r}) = \vec{\nabla}\times\vec{\Lambda}(\vec{r}) = \frac{\mu_0}{4\,\pi}\iiint \frac{\vec{J}(\vec{R})\times(\vec{r} - \vec{R})}{|\,\vec{r} - \vec{R}\,|^3}\;d^3R \tag{274}$$

---

[25] The most general solution of Eq. (267) is given by
$\beta(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\,d^3R\;+\;\beta_{hom}(\vec{r})$
whereby $\beta_{hom}(\vec{r})$ is a solution of the Laplace equation $\Delta\beta_{hom}\,=\,0$

[26] The most general solution of Eq. (268) is given by
$\vec{\Lambda}(\vec{r})\;=\;\frac{\mu_0}{4\,\pi}\iiint \frac{\vec{J}(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\,d^3R\;+\;\vec{\Lambda}_{hom}(\vec{r})$
whereby the components of $\vec{\Lambda}_{hom}(\vec{r})$ are solutions of the Laplace equations $\Delta\,\vec{\Lambda}_{hom}\,=\,0$

are considered as given functions because they depend exclusively on the charge density $\rho(\vec{r})$ or current density $\vec{J}(\vec{r})$.

A solution of Eqs. (271) and (272) for the most general case is presently not known. However, solutions are presented for the two special cases $\vec{B}(\vec{r}) = 0$ as well as $\vec{B}(\vec{r}) = \vec{E}(\vec{r}) = 0$, see section 2.5.1, especially Eqs. (255) $-$ (262), and section 2.4. The latter case and its solutions represent possible level II vacuum potentials and fields in the absence of level I fields.

The scalar potential $\phi(\vec{r})$ and vector potential $\vec{A}(\vec{r})$ in the original equations (77) $-$ (82) and transformed equations (271) and (272) is not identical with the scalar potential $\beta(\vec{r})$ and vector potential $\vec{\Lambda}(\vec{r})$ of textbook electro- and magnetostatics, respectively. $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ are rather part of another set of quantities, namely the level II potentials and fields

$$\phi(\vec{r}) \ , \ g(\vec{r}) \ , \ \vec{E}_\phi(\vec{r}) \ , \ \vec{E}_g(\vec{r}) \ , \ \vec{A}(\vec{r}) \ , \ \vec{V}(\vec{r}) \ , \ \vec{B}_A(\vec{r}) \ , \ \vec{B}_V(\vec{r})$$

which describe a possible physical reality beyond that of textbook electro- and magnetostatics.

The level I potentials $\beta(\vec{r})$ and $\vec{\Lambda}(\vec{r})$ are determined by the decoupled linear differential equations (267) and (268), whereas the level II potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ are specified by the coupled non-linear differential equations (271) and (272). Thus for the level I potentials and fields there is no coupling between electric and magnetic quantities and features, whereas for the level II potentials and fields there is a coupling between electric and magnetic quantities and features. We raise the question if the level II potentials and fields, or their effects, are physically detectable (with present technology) and if they carry the same physcial qualities like the level I potentials and fields.

3 The latest set of the electro- and magnetostatic ECE equations

3.1 The set of equations in its original form

Recent ECE papers report on the discovery of additional equations, the so-called antisymmetry constraints [6, 23, 24]. These additional equations lead to a modification of the electrodynamic as well as electro- and magnetostatic ECE equations [6, 23, 24].

The latest set of the electro- and magnetostatic ECE equations, see e.g. Ref. [22], is given by the former equations (77) $-$ (82) and the so-called antisymmetry constraints, namely [27]

[27] Eqs. (275) $-$ (284) refer to the assumption that the so-called polarization index can be omitted, i.e. one polarization only, see e.g. Ref. [22].

Field equations in terms of potentials:

$$\vec{\nabla} \cdot \left(\vec{\omega} \times \vec{A} \right) \; = \; 0 \tag{275}$$

$$\vec{\nabla} \times \left(\vec{\omega} \, \phi \; - \; \omega_0 \, \vec{A} \right) \; = \; 0 \tag{276}$$

$$\Delta \phi \; - \; \vec{\nabla} \cdot \left(\vec{\omega} \, \phi \; - \; \omega_0 \, \vec{A} \right) \; = \; - \frac{\rho}{\epsilon_0} \tag{277}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} - \vec{\omega} \times \vec{A} \right) \; = \; \mu_0 \; \vec{J} \tag{278}$$

Antisymmetry constraints:

$$\vec{\nabla} \phi \; = \vec{\omega} \, \phi \; + \; \omega_0 \, \vec{A} \tag{279}$$

$$\frac{\partial A_3}{\partial \, y} + \frac{\partial A_2}{\partial \, z} + \omega_2 A_3 + \omega_3 A_2 = 0 \tag{280}$$

$$\frac{\partial A_3}{\partial \, x} + \frac{\partial A_1}{\partial \, z} + \omega_1 A_3 + \omega_3 A_1 = 0 \tag{281}$$

$$\frac{\partial A_2}{\partial \, x} + \frac{\partial A_1}{\partial \, y} + \omega_1 A_2 + \omega_2 A_1 = 0 \tag{282}$$

whereby $\vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ and $\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$

Field-potential relations:

$$\vec{B} \; = \; \vec{\nabla} \times \vec{A} \; - \; \vec{\omega} \times \vec{A} \tag{283}$$

$$\vec{E} = - \; \vec{\nabla} \phi + \vec{\omega} \; \phi \; - \; \omega_0 \, \vec{A} \tag{284}$$

The electric field $\vec{E}$ can also be represented in another way. By inserting Eq. (279) into Eq. (284) we get

$$\vec{E} = - \, 2 \, \omega_0 \, \vec{A} \tag{285}$$

Another expression for the electric field $\vec{E}$ can be obtained by solving Eq. (279) for $\omega_0 \vec{A}$ and inserting it into Eq. (284). This leads to

$$\vec{E} = - \, 2 \, \vec{\nabla} \phi \; + \; 2 \, \vec{\omega} \; \phi \tag{286}$$

Eqs. (275) − (284) merge into the equations of textbook electro- and magnetostatics if the antisymmetry constraints (279) − (282) are omitted and

- $\vec{\omega}\,\phi = \vec{A}\,\omega_0$

  which implies $\vec{\omega} \times \vec{A} = 0$ because in this case $\vec{\omega}$ and $\vec{A}$ are (anti)parallel to each other − this condition is reported in Ref [13] for the electrodynamic equations and works also for the electro- and magnetostatic case

  or

- $\vec{\omega} = 0$ and $\omega_0 = 0$

Eqs. (275) − (282) represent 14 equations to determine the 8 quantities ϕ , $\vec{A}$, ω_0 and $\vec{\omega}$, i.e. there are more equations than variables. In the following we will consider and investigate these equations in more detail.

3.2 Results and discussion of the transformed equations

Eqs. (280) − (282) mean that the vector spin connection $\vec{\omega}$ is entirely specified by the vector potential $\vec{A}$. They can be solved for $\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ and by somewhat algebra we obtain

$$\boxed{\begin{aligned}
\omega_1 &= \frac{1}{2A_2A_3}\left(A_1\frac{\partial A_3}{\partial y} + A_1\frac{\partial A_2}{\partial z} - A_2\frac{\partial A_3}{\partial x} - A_2\frac{\partial A_1}{\partial z} - A_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial A_1}{\partial y}\right) \quad (287)\\[2ex]
\omega_2 &= \frac{1}{2A_1A_3}\left(-A_1\frac{\partial A_3}{\partial y} - A_1\frac{\partial A_2}{\partial z} + A_2\frac{\partial A_3}{\partial x} + A_2\frac{\partial A_1}{\partial z} - A_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial A_1}{\partial y}\right) \quad (288)\\[2ex]
\omega_3 &= \frac{1}{2A_1A_2}\left(-A_1\frac{\partial A_3}{\partial y} - A_1\frac{\partial A_2}{\partial z} - A_2\frac{\partial A_3}{\partial x} - A_2\frac{\partial A_1}{\partial z} + A_3\frac{\partial A_2}{\partial x} + A_3\frac{\partial A_1}{\partial y}\right) \quad (289)
\end{aligned}}$$

In Eqs. (275), (278), and (283) appears the vector product $\vec{\omega} \times \vec{A}$. By means of Eqs. (287) − (289) we can express the vector product

$$\vec{\omega} \times \vec{A} = \begin{pmatrix} \omega_2\,A_3 - \omega_3\,A_2 \\ \omega_3\,A_1 - \omega_1\,A_3 \\ \omega_1\,A_2 - \omega_2\,A_1 \end{pmatrix} \tag{290}$$

in terms of $\vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ and get

$$
\vec{\omega} \times \vec{A} = \begin{pmatrix} \frac{A_2}{A_1}\frac{\partial A_3}{\partial x} + \frac{A_2}{A_1}\frac{\partial A_1}{\partial z} - \frac{A_3}{A_1}\frac{\partial A_2}{\partial x} - \frac{A_3}{A_1}\frac{\partial A_1}{\partial y} \\ \\ \frac{A_3}{A_2}\frac{\partial A_2}{\partial x} + \frac{A_3}{A_2}\frac{\partial A_1}{\partial y} - \frac{A_1}{A_2}\frac{\partial A_3}{\partial y} - \frac{A_1}{A_2}\frac{\partial A_2}{\partial z} \\ \\ \frac{A_1}{A_3}\frac{\partial A_3}{\partial y} + \frac{A_1}{A_3}\frac{\partial A_2}{\partial z} - \frac{A_2}{A_3}\frac{\partial A_3}{\partial x} - \frac{A_2}{A_3}\frac{\partial A_1}{\partial z} \end{pmatrix} \tag{291}
$$

By inserting Eq. (283) into Eq. (278) we obtain

$$
\vec{\nabla} \times \vec{B} = \mu_0 \, \vec{J} \tag{292}
$$

which is the same relation between magnetic field $\vec{B}$ and current density $\vec{J}$ as in textbook magneto- and electrostatics. From Eq. (275) we infer that $\vec{\omega} \times \vec{A}$ can be written as

$$
\vec{\omega} \times \vec{A} = \vec{\nabla} \times \vec{V} \tag{293}
$$

whereby $\vec{V} = \vec{V}(\vec{r})$ is another vector potential which depends in some way on the vector potential $\vec{A} = \vec{A}(\vec{r})$ [28].

We introduce another vector potential $\vec{\Lambda} = \vec{\Lambda}(\vec{r})$ by

$$
\vec{\Lambda} = \vec{A} - \vec{V} \tag{294}
$$

From Eqs. (283), (293), and (294) it follows that the magnetic field $\vec{B}$ can be represented as

$$
\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{\Lambda} = \vec{\nabla} \times \left(\vec{A} - \vec{V}\right) = \vec{\nabla} \times \vec{A} - \vec{\nabla} \times \vec{V} \\
&= \vec{\nabla} \times \vec{A} - \vec{\omega} \times \vec{A}
\end{aligned} \tag{295}
$$

With respect to the magnetic field $\vec{B}$, current density $\vec{J}$ and vector potential $\vec{\Lambda}$, the Eqs. (292) and (295) are the same as those in textbook magneto- and electrostatics. Thus $\vec{\Lambda}$ coresponds to the vector potential used in textbook magneto- and electrostatics. By inserting Eq. (295) into Eq. (292) and by means of Eq. (83) we get

$$
\vec{\nabla} \times (\vec{\nabla} \times \vec{\Lambda}) = \vec{\nabla}\,(\vec{\nabla} \cdot \vec{\Lambda}) - \Delta\,\vec{\Lambda} = \mu_0\,\vec{J} \tag{296}
$$

[28] Actually, from Eqs. (291), (293), (51) and (52) it is possible to determine how $\vec{V}(\vec{r})$ depends on $\vec{A}(\vec{r})$. Equation (291) tells us how $\vec{\omega}(\vec{r}) \times \vec{A}(\vec{r})$ depends on $\vec{A}(\vec{r})$. Let's call the right-hand side of Eq. (291) as $\vec{G}\left(\vec{A}(\vec{r})\right)$. Then from Eqs. (291) and (293) we obtain $\vec{\nabla} \times \vec{V}(\vec{r}) = \vec{G}\left(\vec{A}(\vec{r})\right)$. By using this relation it follows from Eqs. (51) and (52) that $\vec{V}(\vec{r})$ is given by $\vec{V}(\vec{r}) = -\,\vec{r} \times \left[\int_0^1 \vec{G}\left(\vec{A}(s\,\vec{r})\right) s\,ds\right]$

By using the so-called Coulomb gauge [29], i.e. $\vec{\nabla} \cdot \vec{\Lambda} = 0$, Eq. (296) becomes

$$\Delta \vec{\Lambda} = -\, \mu_0 \, \vec{J} \tag{297}$$

For a spatially limited current density $\vec{J}(\vec{r})$, i.e. $\vec{J}(\vec{r}) \to 0$ for $r \to \infty$, the vector potential $\vec{\Lambda} = \vec{A} - \vec{V}$ which solves Eq. (297) is well-known, namely [30]

$$\vec{\Lambda}(\vec{r}) \;=\; \vec{A}(\vec{r}) - \vec{V}(\vec{r}) \;=\; \frac{\mu_0}{4\,\pi} \iiint \frac{\vec{J}(\vec{r})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \tag{298}$$

By inserting Eq. (298) into Eq. (295) the magnetic field $\vec{B}$ results in

$$\begin{aligned} \vec{B}(\vec{r}) \;&=\; \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r}) \;=\; \vec{\nabla} \times \left[\vec{A}(\vec{r}) - \vec{V}(\vec{r})\right] \;=\; \vec{\nabla} \times \vec{\Lambda}(\vec{r}) \\ &=\; \frac{\mu_0}{4\,\pi} \iiint \frac{\vec{J}(\vec{R}) \times (\vec{r} - \vec{R})}{|\,\vec{r} - \vec{R}\,|^3}\, d^3R \end{aligned} \tag{299}$$

whereby we have introduced the magnetic fields $\vec{B}_A$ and $\vec{B}_V$ by

$$\vec{B}_A = \vec{\nabla} \times \vec{A} \tag{300}$$

$$\vec{B}_V = \vec{\nabla} \times \vec{V} = \vec{\omega} \times \vec{A} \tag{301}$$

According to the integral term in Eq. (298) and (299) the vector potential $\vec{\Lambda}(\vec{r})$ and magnetic magnetic field $\vec{B}(\vec{r})$ is identical with the vector potential and magnetic field of textbook magneto- and electrostatics, respectively. Thus, on the level of the

- vector potential $\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r})$ and

[29] The quantity $\vec{B} = \vec{\nabla} \times \vec{\Lambda}$ and the equation $\vec{\nabla} \times \vec{\nabla} \times \vec{\Lambda} = \vec{J}$ remains invariant under a so-called gauge transformation $\vec{\Lambda} \to \vec{\Gamma} = \vec{\Lambda} + \vec{\nabla}\,\eta$ whereby η is any scalar field. By using Eq. (83) we can write $\vec{\nabla} \times \vec{\nabla} \times \left(\vec{\Lambda} + \vec{\nabla}\eta\right) = \vec{J}$ as
$\vec{\nabla}\left(\vec{\nabla} \cdot \left(\vec{\Lambda} + \vec{\nabla}\eta\right)\right) - \Delta\left(\vec{\Lambda} + \vec{\nabla}\eta\right) = \vec{J}$. Assuming that $\vec{\nabla} \cdot \left(\vec{\Lambda} + \vec{\nabla}\eta\right) = 0$, which is always possible by using an appropriate function η , then we get
$\Delta\left(\vec{\Lambda} + \vec{\nabla}\eta\right) = \Delta\,\vec{\Gamma} = -\,\vec{J}$. For a spatially limited current density $\vec{J}$,
i.e. $\vec{J}(\vec{r}) \to 0$ for $r \to \infty$, the well-known solution $\vec{\Gamma}$ of this Poisson equation is given by
$\vec{\Gamma}(\vec{r}) = \frac{1}{4\,\pi} \iiint \frac{\vec{J}(\vec{r})}{|\,\vec{r} - \vec{R}\,|}\, d^3R$. The choice or gauge $\vec{\nabla} \cdot \left(\vec{\Lambda} + \vec{\nabla}\eta\right) = \vec{\nabla} \cdot \vec{\Gamma} = 0$ is usually called Coulomb gauge.

[30] The most general solution of Eq. (297) is given by
$\vec{\Lambda}(\vec{r}) = \frac{\mu_0}{4\,\pi} \iiint \frac{\vec{J}(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\, d^3R \,+\, \vec{\Lambda}_{hom}(\vec{r})$
whereby the components of $\vec{\Lambda}_{hom}(\vec{r})$ are solutions of the Laplace equations $\Delta\,\vec{\Lambda}_{hom} = 0$

- magnetic field $\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r})$

textbook and ECE magneto- and electrostatics are equivalent. However, in ECE Theory

- the vector potential $\vec{\Lambda}(\vec{r})$ emerges from a difference between two vector potentials whose curl do not vanish
- the magnetic field $\vec{B}(\vec{r})$ emerges from a difference between two magnetic fields which both depend on $\vec{r}$

whereas in textbook magneto- and electrostatics they result always from one spatially dependent function.

The so-called level II quantities, namely the vector potentials $\vec{A}(\vec{r})$ and $\vec{V}(\vec{r})$ and magnetic fields $\vec{B}_A(\vec{r})$ and $\vec{B}_V(\vec{r})$, point to the existence of a possible physical reality beyond that of textbook magneto- and electrostatics. We raise the questions if these level II potentials and fields are physically measurable and if they cause the same physcial effects like the level I quantities $\vec{\Lambda}(\vec{r})$ and $\vec{B}(\vec{r})$. Possibly, the level II potentials and fields, or their effects, are physically not detectable (with present technology) or their experimental verification requires special circumstances.

We emphasize that ECE and textbook magneto- and electrostatics usually use the same symbol $\vec{A}$ for the vector potential. However, according to the just mentioned considerations, which are based on a view suggested by Eqs. (298) and (299), they do not represent the same quantity and have to be distinguished. In this paper $\vec{A}$ represents the vector potential which appears in the original magneto- and electrostatic ECE equations (279) $-$ (284) and $\vec{\Lambda}$ corresponds to the vector potential of textbook magneto- and electrostatics.

According to Eqs. (294), (295), (299) $-$ (301) the case of

$$\vec{V}(\vec{r}) = \vec{\nabla}\alpha(\vec{r}) \tag{302}$$

, whereby $\alpha(\vec{r})$ is any scalar field, implies

$$\vec{\nabla} \times \vec{V}(\vec{r}) = 0 \tag{303}$$

$$\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{\nabla}\alpha(\vec{r}) \tag{304}$$

$$\vec{B}_V(\vec{r}) = 0 \tag{305}$$

$$\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = \vec{\nabla} \times \vec{\Lambda}(\vec{r}) \tag{306}$$

and corresponds to the scenario of textbook magneto- and electrostatics.

Now let's consider the equations which are associated with the electric field $\vec{E}$ and the scalar potential ϕ. By taking the curl of Eq. (279) we get

$$\vec{\nabla} \times (\vec{\omega}\,\phi\, + \omega_0 \vec{A}) = 0 \tag{307}$$

By comparing Eq. (307) with Eq. (276) we infer that

$$\vec{\nabla} \times (\vec{\omega}\,\phi) = 0 \tag{308}$$

$$\vec{\nabla} \times (\omega_0 \vec{A}) = 0 \tag{309}$$

and thus

$$\vec{\omega}\,\phi = \vec{\nabla} g \tag{310}$$

$$\omega_0 \vec{A} = \frac{1}{2}\,\vec{\nabla}\beta \tag{311}$$

whereby $g = g(\vec{r}\,)$ and $\beta = \beta(\vec{r}\,)$ are scalar fields. The factor $\frac{1}{2}$ in Eq. (311) represents just a convenience which appears useful for the later resulting expression of β .

Eqs. (308) and (309) might involve some difficulties. Because the vector potential $\vec{A}$ and the vector spin connection $\vec{\omega}$ are specified by Eqs. (331) $-$ (333) and (287) $-$ (289), respectively, Eqs. (308) and (309) indicate that the two remaining variables ϕ and ω_0 are determined by them. However, for a given $\vec{\omega}$ and $\vec{A}$ the Eq. (308) and (309) represent three equations to determine the scalar potential ϕ and the scalar spin connection ω_0 , respectively. Therefore it seems not clear if there is always a scalar field ϕ and ω_0 which satisfies Eq. (308) and (309), respectively. Moreover, assuming that ϕ can be determined from Eq. (308), then it appears implausible that Eq. (277), which also comprises the scalar potential ϕ, is not involved in its determination. We can circumvent these possible difficulties in the following way. By inserting Eq. (311) into Eq. (279) we get

$$\vec{\nabla}\phi = \vec{\omega}\,\phi\, + \frac{1}{2}\,\vec{\nabla}\beta \tag{312}$$

By taking the divergence on both sides of Eq. (312) and using $\vec{\nabla} \cdot \vec{\nabla} = \Delta$ we obtain

$$\vec{\nabla} \cdot (\vec{\omega}\,\phi) = \;\Delta\phi - \frac{1}{2}\Delta\beta \tag{313}$$

Inserting Eqs. (313) and (311) into Eq. (277) leads to

$$\Delta\phi - (\Delta\phi - \frac{1}{2}\Delta\beta)\; + \frac{1}{2}\,\vec{\nabla} \cdot (\vec{\nabla}\beta) = -\frac{\rho}{\epsilon_0} \tag{314}$$

and by means of $\vec{\nabla} \cdot \vec{\nabla} = \Delta$ this results in

$$\Delta\beta = -\frac{\rho}{\epsilon_0} \tag{315}$$

The insertion of Eq. (310) into Eq. (312) reveals that $\beta(\vec{r})$ represents a scalar potential which emerges from the difference of the scalar potentials $\phi(\vec{r})$ and $g(\vec{r})$, i.e.

$$\frac{1}{2}\,\vec{\nabla}\beta = \vec{\nabla}\,(\phi - g) \tag{316}$$

and thus

$$\frac{1}{2}\,\beta(\vec{r}) = \phi_0 + \phi(\vec{r}) - g(\vec{r}) \tag{317}$$

whereby ϕ_0 is a constant. For the sake of simplicity we choose $\phi_0 = 0$ and thus

$$\beta(\vec{r}) = 2\left[\phi(\vec{r}) - g(\vec{r})\right] \tag{318}$$

For a spatially limited charge density $\rho(\vec{r})$, i.e. $\rho(\vec{r}) \to 0$ for $r \to \infty$, the solution $\beta(\vec{r})$ of Eq. (315) is well-known and by taking into account Eq. (318) we obtain [31]

$$\boxed{\beta(\vec{r}) = 2\left[\phi(\vec{r}) - g(\vec{r})\right] = \frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\,d^3R} \tag{319}$$

By inserting Eq. (319) into Eq. (311) we get

$$\omega_0\,\vec{A} = \frac{1}{2}\,\vec{\nabla}\beta = \vec{\nabla}\,(\phi - g) = -\,\frac{1}{2}\,\frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{(\vec{r} - \vec{R})\,\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|^3}\,d^3R \tag{320}$$

The electric field $\vec{E} = \vec{E}(\vec{r})$ results from an insertion of Eq. (320) into Eq. (285), namely

$$\begin{aligned}\vec{E}(\vec{r}) &= -\,2\,\omega_0(\vec{r})\,\vec{A}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r}) = -\,\vec{\nabla}\beta(\vec{r}) \\ &= -\,2\,\vec{\nabla}\left[\phi(\vec{r})\;-\;g(\vec{r})\right] = \frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{(\vec{r} - \vec{R})\,\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|^3}\,d^3R\end{aligned} \tag{321}$$

whereby we have introduced the electric fields $\vec{E}_\phi$ and $\vec{E}_g$ by

$$\vec{E}_\phi = -2\,\vec{\nabla}\phi \tag{322}$$

$$\vec{E}_g = -2\,\vec{\nabla}g \tag{323}$$

[31] The most general solution of Eq. (315) is given by
$\beta(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\iiint \frac{\rho(\vec{R})}{|\,\vec{r} - \vec{R}\,|}\,d^3R \;+\; \beta_{hom}(\vec{r})$
whereby $\beta_{hom}(\vec{r})$ is a solution of the Laplace equation $\Delta\beta_{hom} = 0$

According to the intergral term in Eq. (319) and (321) the scalar potential $\beta(\vec{r})$ and electric field $\vec{E}(\vec{r})$ is identical with the scalar potential and electric field of textbook electro- and magnetostatics, respectively. Thus, on the level of the scalar potential $\beta(\vec{r})$ and electric field $\vec{E}(\vec{r})$ textbook and ECE electro- and magnetostatics are equivalent. However, in ECE electro- and magnetostatics the scalar potential $\beta(\vec{r})$ and electric field $\vec{E}(\vec{r})$ emerge from a difference between two quantities which both depend on $\vec{r}$, whereas in textbook electro- and magnetostatics they result from always one spatially dependent function.

The so-called level II quantities, namely the scalar potentials $\phi(\vec{r})$ and $g(\vec{r})$ and the electric fields $\vec{E}_\phi(\vec{r})$ and $\vec{E}_g(\vec{r})$, point to the existence of a possible physical reality beyond that of textbook magneto- and electrostatics. We raise the questions if these level II potentials and fields are physically measurable and if they cause the same physcial effects like the level I quantities $\beta(\vec{r})$ and $\vec{E}(\vec{r})$. Possibly, the level II potentials and fields, or their effects, are physically not detectable (with present technology) or their experimental verification requires special circumstances.

We emphasize that ECE and textbook electro- and magnetostatics usually use the same symbol ϕ for the scalar potential. However, according to the just mentioned considerations, which are based on a view suggested by Eqs. (319) and (321), they do not represent the same quantity and have to be distinguished. In this paper ϕ represents the scalar potential which appears in the original electro- and magnetostatic ECE equations (279) $-$ (284) and β corresponds to the scalar potential of textbook electro- and magnetostatics.

According to Eqs. (319) and (321) $-$ (323) the case of

$$g(\vec{r}) = g_0 \tag{324}$$

, whereby g_0 is a constant, implies

$$\beta(\vec{r}) = 2\,\phi(\vec{r}) - 2\,g_0 \tag{325}$$

$$\vec{E}_g(\vec{r}) = 0 \tag{326}$$

$$\vec{E}(\vec{r}) = -\,\vec{\nabla}\beta(\vec{r}) = -2\,\vec{\nabla}\phi(\vec{r}) \tag{327}$$

and corresponds to the scenario of textbook magneto- and electrostatics.

3.3 The equations for the level II potentials

Eqs. (299) and (291) can be used to establish a set of non-linear, first-order differential equations for the vector potential $\vec{A}$. By inserting Eq. (291) into Eq. (283) we get

$$A_1 B_1 = A_1\left(\frac{\partial A_3}{\partial\,y} - \frac{\partial A_2}{\partial\,z}\right) - A_2\left(\frac{\partial A_3}{\partial\,x} + \frac{\partial A_1}{\partial\,z}\right) + A_3\left(\frac{\partial A_2}{\partial\,x} + \frac{\partial A_1}{\partial\,y}\right) \tag{328}$$

$$A_2 B_2 = A_2\left(\frac{\partial A_1}{\partial\, z} - \frac{\partial A_3}{\partial\, x}\right) - A_3\left(\frac{\partial A_2}{\partial\, x} + \frac{\partial A_1}{\partial\, y}\right) + A_1\left(\frac{\partial A_3}{\partial\, y} + \frac{\partial A_2}{\partial\, z}\right) \tag{329}$$

$$A_3 B_3 = A_3\left(\frac{\partial A_2}{\partial\, x} - \frac{\partial A_1}{\partial\, y}\right) - A_1\left(\frac{\partial A_3}{\partial\, y} + \frac{\partial A_2}{\partial\, z}\right) + A_2\left(\frac{\partial A_3}{\partial\, x} + \frac{\partial A_1}{\partial\, z}\right) \tag{330}$$

By adding Eqs. (329) and (330), Eqs. (328) and (329), and Eqs. (328) and (330), respectively, we obtain

$$2 A_2 \frac{\partial A_1}{\partial\, z} - 2 A_3 \frac{\partial A_1}{\partial\, y} - A_2 B_2 - A_3 B_3 = 0 \tag{331}$$

$$2 A_1 \frac{\partial A_3}{\partial\, y} - 2 A_2 \frac{\partial A_3}{\partial\, x} - A_1 B_1 - A_2 B_2 = 0 \tag{332}$$

$$2 A_3 \frac{\partial A_2}{\partial\, x} - 2 A_1 \frac{\partial A_2}{\partial\, z} - A_1 B_1 - A_3 B_3 = 0 \tag{333}$$

Furthermore, the vector potential

$$\vec{A}(\vec{r}) = \begin{pmatrix} A_1(\vec{r}) \\ A_2(\vec{r}) \\ A_3(\vec{r}) \end{pmatrix}$$

which solves Eqs. (331) $-$ (333) must additionally satisfy Eq. (275) which is via Eq. (291) given by

$$\begin{aligned} \vec{\nabla}\cdot\left(\vec{\omega}\times\vec{A}\right) = \; & \frac{\partial}{\partial\, x}\left(\frac{A_2}{A_1}\frac{\partial A_3}{\partial\, x} + \frac{A_2}{A_1}\frac{\partial A_1}{\partial\, z} - \frac{A_3}{A_1}\frac{\partial A_2}{\partial\, x} - \frac{A_3}{A_1}\frac{\partial A_1}{\partial\, y}\right) \\ & + \frac{\partial}{\partial\, y}\left(\frac{A_3}{A_2}\frac{\partial A_2}{\partial\, x} + \frac{A_3}{A_2}\frac{\partial A_1}{\partial\, y} - \frac{A_1}{A_2}\frac{\partial A_3}{\partial\, y} - \frac{A_1}{A_2}\frac{\partial A_2}{\partial\, z}\right) \\ & + \frac{\partial}{\partial\, z}\left(\frac{A_1}{A_3}\frac{\partial A_3}{\partial\, y} + \frac{A_1}{A_3}\frac{\partial A_2}{\partial\, z} - \frac{A_2}{A_3}\frac{\partial A_3}{\partial\, x} - \frac{A_2}{A_3}\frac{\partial A_1}{\partial\, z}\right) = 0 \end{aligned} \tag{334}$$

In Eqs. (331) $-$ (333) the level I magnetic field

$$\vec{B}(\vec{r}) = \begin{pmatrix} B_1(\vec{r}) \\ B_2(\vec{r}) \\ B_3(\vec{r}) \end{pmatrix} = \frac{\mu_0}{4\,\pi}\iiint \frac{\vec{J}(\vec{R})\times(\vec{r}-\vec{R})}{|\,\vec{r}-\vec{R}\,|^3}\, d^3R \tag{335}$$

(see Eq. (299)) is considered as a given function because it depends exclusively on the current density $\vec{J}(\vec{r})$. Eqs. (331) $-$ (334) are four equations for the three components $A_1(\vec{r})$, $A_2(\vec{r})$ and $A_3(\vec{r})$ of the level II vector potential $\vec{A}(\vec{r})$. Thus $\vec{A}(\vec{r})$ is possibly over-determined and we raise the question if always a solution exits. Further studies are necessary to clarify this issue.

We note that the Eqs. (331) − (334), which specify the level II vector potential $\vec{A}(\vec{r})$, are four first-order non-linear coupled differential equations, whereas Eq. (297), which determines the level I vector potential $\vec{\Lambda}(\vec{r})$, represents three second-order linear decoupled differential equations.

Once the level II vector potential

$$\vec{A}(\vec{r}) = \begin{pmatrix} A_1(\vec{r}) \\ A_2(\vec{r}) \\ A_3(\vec{r}) \end{pmatrix}$$

is determined from Eqs. (331) − (334), the associated level II quantities $\vec{V}(\vec{r})$, $\vec{B}_A(\vec{r})$ and $\vec{B}_V(\vec{r})$ can be computed from Eqs. (298), (300) and (301).

In the following we present several forms of equations which specify the level II scalar potential $\phi(\vec{r})$.

The scalar potential $\phi(\vec{r})$ can be determined by Eq. (286), namely

$$-2\,\vec{\nabla}\phi + 2\,\phi\,\vec{\omega} = \vec{E} \tag{336}$$

whereby

- the vector spin connection $\vec{\omega}(\vec{r})$ is via Eqs. (287) − (289) entirely specified by the vector potential $\vec{A}(\vec{r})$ which is in turn determined by Eqs. (331) − (334)
- the electric field $\vec{E}(\vec{r})$, which is given by Eq. (321), is considered as a given function because it depends exclusively on the charge density $\rho(\vec{r})$

Eq. (336) comprises three equations for the specification of $\phi(\vec{r})$. Thus $\phi(\vec{r})$ is possibly over-determined and we raise the question if always a solution exits. However, by taking into account Eqs. (310) and (321) we realize that each of the three terms of Eq. (336) is a gradient of a scalar function. This feature probably favors the existence of solutions. Nevertheless, it remains an open question if $\vec{\omega}(\vec{r})$ which is specified by $\vec{A}(\vec{r})$ via Eqs. (287) − (289) and (331) − (334) always ensures the existence of a solution of Eq. (336).

There are another forms of equations for $\phi(\vec{r})$. By means of

$$\vec{\omega}(\vec{r}) = \begin{pmatrix} \omega_1(\vec{r}) \\ \omega_2(\vec{r}) \\ \omega_3(\vec{r}) \end{pmatrix} \tag{337}$$

$$\vec{E}(\vec{r}) = \begin{pmatrix} E_1(\vec{r}) \\ E_2(\vec{r}) \\ E_3(\vec{r}) \end{pmatrix} \tag{338}$$

Eq. (336) can be written as

$$- 2 \frac{\partial \phi}{\partial x} + 2 \phi \omega_1 = E_1 \tag{339}$$

$$- 2 \frac{\partial \phi}{\partial y} + 2 \phi \omega_2 = E_2 \tag{340}$$

$$- 2 \frac{\partial \phi}{\partial z} + 2 \phi \omega_3 = E_3 \tag{341}$$

These are first-order differential equations of the type

$$\frac{d \phi}{d x_i} + u_i(x, y, z) \phi(x, y, z) = v_i(x, y, z) \tag{342}$$

whereby

$$x_1 = x \quad x_2 = y \quad x_3 = z \qquad i = 1, 2, 3$$

$$u_i(x, y, z) = - \omega_i(x, y, z) \tag{343}$$

$$v_i(x, y, z) = - \frac{E_i(x, y, z)}{2} \tag{344}$$

The solution of Eq. (339) is well-known, namely

$$\begin{aligned} &\phi(x, y, z) = \\ &\exp\left(- \int u_i(x, y, z) \, dx_i\right) \left[\phi_{0\,i} + \int v_i(x, y, z) \, \exp\left(\int u_i(x, y, z) \, dx_i\right) dx_i\right] \end{aligned} \tag{345}$$

whereby $\phi_{0\,i}$ are constants. By inserting Eqs. (343), (344) and (321) and into Eq. (345) we get

$$\phi(\vec{r}) \;=\; \exp\Big(\int \omega_i(\vec{r})\,dx_i\Big)\left[\phi_{0\,i} \;+ \frac{1}{2}\int \frac{\partial\,\beta(\vec{r})}{\partial\,x_i}\;\exp\Big(-\!\int \omega_i(\vec{r})\,dx_i\Big)\,dx_i\right] \tag{346}$$

whereby

$$x_1 = x \quad x_2 = y \quad x_3 = z \qquad\quad i = 1, 2, 3$$

and the three components $\omega_i(\vec{r})$ are specified by the vector potential

$$\vec{A}(\vec{r}) = \begin{pmatrix} A_1(\vec{r}) \\ A_2(\vec{r}) \\ A_3(\vec{r}) \end{pmatrix}$$

via Eqs. (287) – (289). The vector potential $\vec{A}\,(\vec{r})$ is in turn specified by Eqs. (331) – (334). The level I scalar potential $\beta(\vec{r})$, which is given by Eq. (319), is considered as a given function because it depends exclusively on the charge density $\rho(\vec{r})$. Because of $i = 1, 2, 3$ the Eq. (346) comprises three different expressions. It represents a solution only if all of them lead to the same function $\phi(\vec{r})$. It remains an open question if this condition can always be accomplished for those $\vec{\omega}\,(\vec{r})$ which are specified by $\vec{A}(\vec{r})$ via Eqs. (287) – (289) and (331) – (334).

Eq. (346) represents a solution if

$$\omega_i(\vec{r}) = \frac{\partial}{\partial\,x_i}\;\ln\frac{\beta(\vec{r})}{\beta_0} \tag{347}$$

and

$$\phi_{0\,1} = \phi_{0\,2} = \phi_{0\,3} = \phi_0 \tag{348}$$

whereby ϕ_0 and β_0 are constants. Inserting Eqs. (347) and (348) into Eq. (346) yields the same expression for $i = 1, 2, 3$, namely

$$\phi(\vec{r}) \;=\; \frac{\beta(\vec{r})}{\beta_0}\left[\phi_0 \;+ \frac{\beta_0}{2}\ln\frac{\beta(\vec{r})}{\beta_0}\right] \tag{349}$$

In this case $\phi(\vec{r})$ does not depend on $\vec{\omega}\,(\vec{r})$ and $\vec{A}(\vec{r})$ which are determined by Eqs. (287) – (289) and (331) – (334).

Presently we do not know if Eq. (347) is the only way which leads to a solution. By taking into account Eqs. (287) – (289) we see that Eq. (347) involves another contraints for the vector potential $\vec{A}(\vec{r})$, namely

$$\begin{aligned} &\frac{\partial}{\partial\,x}\;\ln\frac{\beta}{\beta_0} = \\ &\frac{1}{2A_2A_3}\left(A_1\frac{\partial A_3}{\partial\,y} + A_1\frac{\partial A_2}{\partial\,z} - A_2\frac{\partial A_3}{\partial\,x} - A_2\frac{\partial A_1}{\partial\,z} - A_3\frac{\partial A_2}{\partial\,x} - A_3\frac{\partial A_1}{\partial\,y}\right) \end{aligned} \tag{350}$$

$$\frac{\partial}{\partial y} \ln \frac{\beta}{\beta_0} = \tag{351}$$
$$\frac{1}{2A_1A_3}\left(-A_1\frac{\partial A_3}{\partial y} - A_1\frac{\partial A_2}{\partial z} + A_2\frac{\partial A_3}{\partial x} + A_2\frac{\partial A_1}{\partial z} - A_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial A_1}{\partial y}\right)$$

$$\frac{\partial}{\partial z} \ln \frac{\beta}{\beta_0} = \tag{352}$$
$$\frac{1}{2A_1A_2}\left(-A_1\frac{\partial A_3}{\partial y} - A_1\frac{\partial A_2}{\partial z} - A_2\frac{\partial A_3}{\partial x} - A_2\frac{\partial A_1}{\partial z} + A_3\frac{\partial A_2}{\partial x} + A_3\frac{\partial A_1}{\partial y}\right)$$

whereby $\beta(\vec{r})$, which is given by Eq. (319), is considered as a given function because it depends exclusively on the charge density $\rho(\vec{r})$. Eqs. (350) – (352) and (331) – (334) represent altogether seven equations for the three components $A_1(\vec{r})$, $A_2(\vec{r})$ and $A_3(\vec{r})$ of the vector potential $\vec{A}(\vec{r})$. Thus $\vec{A}(\vec{r})$ appears over-determined and we raise the question if (always) a solution exists. Further studies are necessary to clarify this issue.

The potential difficulties which are associated with the determination of $\phi(\vec{r})$ can possibly be circumvented by the following approach which results in only one equation for $\phi(\vec{r})$. By taking the divergence of Eq. (336) and by using the relation $\vec{E} = -\vec{\nabla}\beta$ from Eq. (321) we obtain

$$\Delta\phi(\vec{r}) - \vec{\nabla}\cdot\left[\vec{\omega}(\vec{r})\,\phi(\vec{r})\right] = \frac{1}{2}\,\Delta\beta(\vec{r}) \tag{353}$$

Inserting Eq. (315) into Eq. (353) yields

$$\boxed{\Delta\phi(\vec{r}) - \vec{\nabla}\cdot\left[\vec{\omega}(\vec{r})\,\phi(\vec{r})\right] = -\frac{1}{2}\,\frac{\rho(\vec{r})}{\epsilon_0}} \tag{354}$$

whereby the vector spin connection $\vec{\omega}(\vec{r})$ is via Eqs. (287) – (289) entirely specified by the vector potential $\vec{A}(\vec{r})$ which is in turn determined by the current density $\vec{J}(\vec{r})$ via Eqs. (331) – (334). In contrast to Eq. (336), which involves three first-order differential equations, the Eq. (354) represents one second-order differential equation.

Once the level II scalar potential $\phi(\vec{r})$ is computed from Eq. (336), (346) or (354), the associated level II quantities $g(\vec{r})$, $\vec{E}_\phi(\vec{r})$ and $\vec{E}_g(\vec{r})$ can be computed from Eqs. (319), (322) and (323).

3.4 The equations for the level II potentials in the absence of level I fields: The vacuum equations

Recently H. Eckardt and D. W. Lindstrom have published a paper about the solutions of the latest set of electrodynamic ECE equations in the absence of level I electric and magnetic fields [25]. They point to the existence of non-vanishing vacuum potentials [25].

In the following we will study the latest set of electro- and magnetostatic ECE equations in the absence of level I electric and magnetic fields, i.e. for $\vec{B} = \vec{E} = 0$. Their solutions indicate the existence of non-vanishing level II vacuum potentials and fields.

The absence of a charge density ρ and current density $\vec{J}$, i.e. $\rho = 0$ and $\vec{J} = 0$, implies $\vec{E} = \vec{B} = 0$ and Eqs. (331) – (334) and (336) result in

$$A_2 \frac{\partial A_1}{\partial z} - A_3 \frac{\partial A_1}{\partial y} = 0 \tag{355}$$

$$A_1 \frac{\partial A_3}{\partial y} - A_2 \frac{\partial A_3}{\partial x} = 0 \tag{356}$$

$$A_3 \frac{\partial A_2}{\partial x} - A_1 \frac{\partial A_2}{\partial z} = 0 \tag{357}$$

$$\begin{aligned} \vec{\nabla} \cdot \left(\vec{\omega} \times \vec{A} \right) = \; & \frac{\partial}{\partial x} \left(\frac{A_2}{A_1} \frac{\partial A_3}{\partial x} + \frac{A_2}{A_1} \frac{\partial A_1}{\partial z} - \frac{A_3}{A_1} \frac{\partial A_2}{\partial x} - \frac{A_3}{A_1} \frac{\partial A_1}{\partial y} \right) \\ & + \frac{\partial}{\partial y} \left(\frac{A_3}{A_2} \frac{\partial A_2}{\partial x} + \frac{A_3}{A_2} \frac{\partial A_1}{\partial y} - \frac{A_1}{A_2} \frac{\partial A_3}{\partial y} - \frac{A_1}{A_2} \frac{\partial A_2}{\partial z} \right) \\ & + \frac{\partial}{\partial z} \left(\frac{A_1}{A_3} \frac{\partial A_3}{\partial y} + \frac{A_1}{A_3} \frac{\partial A_2}{\partial z} - \frac{A_2}{A_3} \frac{\partial A_3}{\partial x} - \frac{A_2}{A_3} \frac{\partial A_1}{\partial z} \right) = 0 \end{aligned} \tag{358}$$

$$\vec{\nabla} \phi = \phi \, \vec{\omega} \tag{359}$$

whereby the vector spin connection $\vec{\omega}(\vec{r})$ is via Eqs. (287) – (289) entirely specified by the vector potential $\vec{A}(\vec{r})$ which is in turn determined by Eqs. (355) – (357).

Eqs. (355) – (359) specify the level II vacuum potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ in the absence of level I fields.

We note that the coupled non-linear first-order differential equations (355) – (357) can also be written in a shortened notation, namely

$$A_i \frac{\partial A_k}{\partial x_j} - A_j \frac{\partial A_k}{\partial x_i} = 0 \tag{360}$$

whereby

- $x_1 = x \quad x_2 = y \quad x_3 = z$
- $i = 1, 2, 3 \quad j = 1, 2, 3 \quad k = 1, 2, 3$

From Eqs. (298) $-$ (301), (319) and (321) $-$ (323) we infer for $\rho = 0$ and $\vec{J} = \vec{E} = \vec{B} = 0$ for the level II potentials and fields the following relations which are associated with Eqs. (355) $-$ (359) and their solutions:

$$g(\vec{r}) = \phi(\vec{r}) \tag{361}$$

$$\vec{E}_\phi(\vec{r}) = \vec{E}_g(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) \tag{362}$$

$$\vec{V}(\vec{r}) = \vec{A}(\vec{r}) \tag{363}$$

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) \tag{364}$$

We recall that these relations mean that all level I potentials and fields vanish, i.e.

$$\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = 0 \tag{365}$$

$$\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r}) = 0 \tag{366}$$

$$\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r}) = 0 \tag{367}$$

$$\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r}) = 0 \tag{368}$$

3.4.1 A class of general solutions

The type of solutions of Eqs. (355) $-$ (357) published in Ref. [25] indicate that a general class of solution is given by

$$\vec{A}(\vec{r}) = \begin{pmatrix} A_1(\vec{r}) \\ A_2(\vec{r}) \\ A_3(\vec{r}) \end{pmatrix} = F(\vec{k}\cdot\vec{r})\ \vec{k} \tag{369}$$

whereby $F = F(s)$ is an arbitrary function of $s = \vec{k}\cdot\vec{r}$ and

$$\vec{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \tag{370}$$

a constant vector. Eq. (369) was used as starting point to search for other and/or more general solutions. This resulted in the finding of two types of general solutions of Eqs. (355) $-$ (357) which will be presented in this and the following section.

The first type of general solutions of Eqs. (355) $-$ (357) is given by

$$A_1(x, y, z) = b'_1(x)\ F(\,b_1(x) + b_2(y) + b_3(z)\,) \tag{371}$$

$$A_2(x, y, z) = b'_2(y)\ F(\,b_1(x) + b_2(y) + b_3(z)\,) \tag{372}$$

$$A_3(x,y,z) = b'_3(z)\; F(\, b_1(x) + b_2(y) + b_3(z)\,) \tag{373}$$

or with respect to Eq. (360) in the shortened notation

$$A_i(x_1,x_2,x_3) = b'_i(x_i)\; F(\, b_1(x_1) + b_2(x_2) + b_3(x_3)\,) \tag{374}$$

or in vector representation

$$\begin{aligned} \vec{A}(\vec{r}\,) \;&=\; F\Big(s(\vec{r}\,)\Big)\, \vec{\nabla}\, s(\vec{r}\,) \;=\; F(\, b_1(x) + b_2(y) + b_3(z)\,) \begin{pmatrix} b'_1(x) \\ b'_2(y) \\ b'_3(z) \end{pmatrix} \\ &=\; \vec{\nabla} \int F(s)\, ds \end{aligned} \tag{375}$$

whereby $F(s) = F\Big(s(\vec{r}\,)\Big)$ is any function of

$$s = s(\vec{r}\,) = b_1(x) + b_2(y) + b_3(z) \tag{376}$$

and $b_1(x)$, $b_2(y)$ and $b_3(z)$ are any functions which depend only on x, y and z, respectively, and

$$b'_1(x) = \frac{d\, b_1(x)}{d\, x} \tag{377}$$

$$b'_2(y) = \frac{d\, b_2(y)}{d\, y} \tag{378}$$

$$b'_3(z) = \frac{d\, b_3(z)}{d\, z} \tag{379}$$

By inserting Eqs. (371) $-$ (373) into Eqs. (287) $-$ (289) we obtain the associated vector spin connection $\vec{\omega}$, namely

$$\vec{\omega}\,(\vec{r}\,) = -\frac{F'(\, b_1(x) + b_2(y) + b_3(z)\,)}{F(\, b_1(x) + b_2(y) + b_3(z)\,)} \begin{pmatrix} b'_1(x) \\ b'_2(y) \\ b'_3(z) \end{pmatrix} \tag{380}$$

or

$$\vec{\omega}\,(\vec{r}\,) = -\frac{\vec{\nabla}\, F\Big(s(\vec{r}\,)\Big)}{F\Big(s(\vec{r}\,)\Big)} \tag{381}$$

or

$$\vec{\omega}\,(\vec{r}\,) = -\,\vec{\nabla}\, \ln \left(\frac{F\Big(s(\vec{r}\,)\Big)}{F_0} \right) \tag{382}$$

and by means of Eq. (375)

$$\vec{\omega}(\vec{r}) = -\frac{F'\Big(s(\vec{r})\Big)}{\Big[F\Big(s(\vec{r})\Big)\Big]^2}\,\vec{A}(\vec{r}) \tag{383}$$

whereby F_0 is a constant and

$$F'(s) = \frac{d\,F}{d\,s} \tag{384}$$

Eqs. (383), (382) and (375) imply that $\vec{A}$ and $\vec{\omega}$ are (anti)parallel to each other and curl-free, i.e.

$$\vec{\omega} \times \vec{A} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{\omega} = 0 \tag{385}$$

and thus Eq. (358) is also satisfied.

Now let's consider the scalar potential $\phi(\vec{r})$. By inserting Eq. (382) into Eq. (359) we get

$$\frac{\vec{\nabla}\phi}{\phi} = -\frac{\vec{\nabla}F}{F} \quad \Leftrightarrow \quad \vec{\nabla}\ln\frac{\phi}{\phi_0} = -\,\vec{\nabla}\ln\frac{F}{F_0} \tag{386}$$

whereby ϕ_0 and F_0 are constants and thus

$$\phi\Big(s(\vec{r})\Big) = \frac{\phi_0\,F_0}{F\Big(s(\vec{r})\Big)} \tag{387}$$

The insertion of Eq. (387) into Eqs. (375) reveals the following relation between the vector potential $\vec{A}(\vec{r})$ and scalar potential $\phi(\vec{r})$:

$$\begin{aligned}\vec{A}(\vec{r}) &= \phi_0\,F_0\frac{\vec{\nabla}s(\vec{r})}{\phi\Big(s(\vec{r})\Big)} = \frac{\phi_0\,F_0}{\phi\,(\,b_1(x) + b_2(y) + b_3(z)\,)}\begin{pmatrix} b'_1(x) \\ b'_2(y) \\ b'_3(z)\end{pmatrix} \\ &= \phi_0\,F_0\,\vec{\nabla}\int\frac{ds}{\phi(s)}\end{aligned} \tag{388}$$

whereby the scalar potential

$$\phi(s) = \phi\Big(s(\vec{r})\Big) \quad \text{is any function of} \quad s = s(\vec{r}) = b_1(x) + b_2(y) + b_3(z) \tag{389}$$

and $b_1(x)$, $b_2(y)$ and $b_3(z)$ are any functions which depend only on x, y and z, respectively, and $b'_1(x)$, $b'_2(y)$ and $b'_3(z)$ denote their derivative as defined by Eqs. (377) $-$ (379).

Eq. (389) and (388) represent possible level II vacuum potentials in the absence of level II fields. Please note the validity of Eqs. (361) $-$ (368) in this case.

The insertion of Eq. (386) or (387) into Eqs. (380) $-$ (383) reveals the following relations between the vector spin connection $\vec{\omega}$, scalar potential ϕ and vector potential $\vec{A}$:

$$\vec{\omega}\,(\vec{r}) = \frac{\phi'\,(\,b_1(x)+b_2(y)+b_3(z)\,)}{\phi\,(\,b_1(x)+b_2(y)+b_3(z)\,)}\begin{pmatrix} b'_1(x) \\ b'_2(y) \\ b'_3(z) \end{pmatrix} \tag{390}$$

$$\vec{\omega}\,(\vec{r}) = \vec{\nabla}\,\ln\left(\frac{\phi\left(s(\vec{r})\right)}{\phi_0}\right) \tag{391}$$

and by means of Eq. (388)

$$\vec{\omega}\,(\vec{r}) = \phi'\left(s(\vec{r})\right)\,\vec{A}(\vec{r}) \tag{392}$$

whereby ϕ_0 is a constant and

$$\phi'(s) = \frac{d\,\phi}{d\,s} \tag{393}$$

Eqs. (392), (391) and (388) imply that $\vec{A}$ and $\vec{\omega}$ are (anti)parallel to each other and curl-free, i.e.

$$\vec{\omega}\times\vec{A} \;=\; \vec{\nabla}\times\vec{A} \;=\; \vec{\nabla}\times\vec{\omega} \;=\; 0 \tag{394}$$

and thus Eq. (358) is also satisfied.

3.4.2 Another class of general solutions

Another type of general solutions of Eqs. (355) $-$ (357) is given by

$$A_1(x,y,z) = c'_1(x)c_2(y)c_3(z)\;G(\,c_1(x)c_2(y)c_3(z)\,) \tag{395}$$

$$A_2(x,y,z) = c_1(x)c'_2(y)c_3(z)\;G(\,c_1(x)c_2(y)c_3(z)\,) \tag{396}$$

$$A_3(x,y,z) = c_1(x)c_2(y)c'_3(z)\;G(\,c_1(x)c_2(y)c_3(z)\,) \tag{397}$$

or with respect to Eq. (360) in the shortened notation

$$A_i(x_1,x_2,x_3) = c'_i(x_i)c_j(x_j)c_k(x_k)\;G(\,c_1(x_1)c_2(x_2)c_3(x_3)\,) \tag{398}$$

or in vector representation

$$\vec{A}(\vec{r}) = G\Big(v(\vec{r})\Big)\,\vec{\nabla} v(\vec{r}) = G(\,c_1(x)c_2(y)c_3(z)\,)\begin{pmatrix} c_1'(x)c_2(y)c_3(z) \\ c_1(x)c_2'(y)c_3(z) \\ c_1(x)c_2(y)c_3'(z) \end{pmatrix} \tag{399}$$

$$= \vec{\nabla}\int G(v)\,dv$$

whereby $G(v) = G\Big(v(\vec{r})\Big)$ is any function of

$$v = v(\vec{r}) = c_1(x)c_2(y)c_3(z) \tag{400}$$

and $c_1(x)$, $c_2(y)$ and $c_3(z)$ are any functions which depend only on x, y and z, respectively, and

$$c_1'(x) = \frac{d\,c_1(x)}{d\,x} \tag{401}$$

$$c_2'(y) = \frac{d\,c_2(y)}{d\,y} \tag{402}$$

$$c_3'(z) = \frac{d\,c_3(z)}{d\,z} \tag{403}$$

Eq. (399) and Eqs. (395) – (398) represent possible level II vacuum vector potentials in the absence of level II fields. Please note the validity of Eqs. (361) – (368) in this case.

By inserting Eqs. (395) – (397) into Eq. (291) we find by somewhat algebra

$$\vec{\omega} \times \vec{A} = 0 \tag{404}$$

and thus Eq. $\vec{\omega}(\vec{r})$ and $\vec{A}(\vec{r})$ are (anti)parallel to each other and Eq. (358) is also satisfied.

By inserting Eqs. (395) – (397) into Eqs. (287) – (289) we obtain the associated vector spin connection

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{405}$$

, namely

$$\omega_1(\vec{r}) = -\frac{c_1'(x)}{c_1(x)}\left[1 + c_1'(x)\,c_2(y)\,c_3(z)\,\frac{G'(\,c_1(x)c_2(y)c_3(z)\,)}{G(\,c_1(x)c_2(y)c_3(z)\,)}\right] \tag{406}$$

$$\omega_2(\vec{r}) = -\frac{c_2'(y)}{c_2(y)}\left[1 + c_1(x)\,c_2'(y)\,c_3(z)\,\frac{G'(\,c_1(x)c_2(y)c_3(z)\,)}{G(\,c_1(x)c_2(y)c_3(z)\,)}\right] \tag{407}$$

$$\omega_3(\vec{r}) = -\frac{c_3'(z)}{c_3(z)}\left[1 + c_1(x)\,c_2(y)\,c_3'(z)\,\frac{G'(\,c_1(x)c_2(y)c_3(z)\,)}{G(\,c_1(x)c_2(y)c_3(z)\,)}\right] \tag{408}$$

whereby

$$G'(v) = \frac{dG}{dv} \tag{409}$$

or

$$\omega_1(\vec{r}) = -\frac{\partial \ln(c_1(x)c_2(y)c_3(z))}{\partial x}\left[1 + \frac{\partial}{\partial x}\ln\left(\frac{G(c_1(x)c_2(y)c_3(z))}{G_{01}}\right)\right] \tag{410}$$

$$\omega_2(\vec{r}) = -\frac{\partial \ln(c_1(x)c_2(y)c_3(z))}{\partial y}\left[1 + \frac{\partial}{\partial y}\ln\left(\frac{G(c_1(x)c_2(y)c_3(z))}{G_{02}}\right)\right] \tag{411}$$

$$\omega_3(\vec{r}) = -\frac{\partial \ln(c_1(x)c_2(y)c_3(z))}{\partial z}\left[1 + \frac{\partial}{\partial z}\ln\left(\frac{G(c_1(x)c_2(y)c_3(z))}{G_{03}}\right)\right] \tag{412}$$

whereby G_{01} , G_{02} and G_{03} are constants. This vector spin connection appears mathematically more complicated than that of the type 1 solution which is given by Eqs. (380) – (383). Eqs. (399) and (406) – (408) or (410) – (412) can be used to establish a relation between $\vec{A}(\vec{r})$ and $\vec{\omega}(\vec{r})$.

In the previous section we have established for the type 1 solution a relation between the vector potential $\vec{A}(\vec{r})$ and the scalar potential $\phi(\vec{r})$, namely Eq. (388). One can try to find a related connection between $\vec{A}(\vec{r})$ and $\phi(\vec{r})$ for the type 2 solution, namely by inserting Eqs. (410) – (412) into Eq. (359) which results in

$$\frac{\partial}{\partial x}\ln\left(\frac{\phi}{\phi_{01}}\right) = -\frac{\partial \ln v}{\partial x}\left[1 + \frac{\partial}{\partial x}\ln\left(\frac{G}{G_{01}}\right)\right] \tag{413}$$

$$\frac{\partial}{\partial y}\ln\left(\frac{\phi}{\phi_{02}}\right) = -\frac{\partial \ln v}{\partial y}\left[1 + \frac{\partial}{\partial y}\ln\left(\frac{G}{G_{02}}\right)\right] \tag{414}$$

$$\frac{\partial}{\partial z}\ln\left(\frac{\phi}{\phi_{03}}\right) = -\frac{\partial \ln v}{\partial z}\left[1 + \frac{\partial}{\partial z}\ln\left(\frac{G}{G_{03}}\right)\right] \tag{415}$$

whereby $G = G(v) = G\Big(v(\vec{r})\Big)$ is any function of

$$v = v(\vec{r}) = c_1(x)c_2(y)c_3(z) \tag{416}$$

and $c_1(x)$, $c_2(y)$ and $c_3(z)$ are any functions which depend only on x, y and z, respectively, and ϕ_{01} , ϕ_{02} and ϕ_{03} are constants.

If there is a scalar potential $\phi(\vec{r})$ which simultaneously solves Eqs. (413) – (415), then that $\phi(x, y, z)$ can be used together with Eq. (399) to establish a relation between $\vec{A}(\vec{r})$ and $\phi(\vec{r})$.

3.4.3 Two open questions

In the previous sections 3.4.1 and 3.4.2 we have presented two general types of solutions of Eqs. (355) $-$ (358), namely the level II vacuum vector potentials which are represented by Eqs. (388) and (399). These level II vacuum vector potentials $\vec{A}(\vec{r})$ are curl-free, i.e. their associated level II vacuum magnetic fields $\vec{B}_A(\vec{r})$ and $\vec{B}_V(\vec{r})$ vanish:

$$\vec{B}_A(\vec{r}) \;=\; \vec{B}_V(\vec{r}) \;=\; \vec{\nabla} \times \vec{A} \;=\; 0 \tag{417}$$

In contrast to that, among the solutions of the vacuum equations of the former set of the electro- and magnetostatic ECE equations, see sections 2.4.3 $-$ 2.4.6, there are level II vacuum vector potentials with a non-vanishing curl and thus non-vanishing level II vacuum magnetic fields.

We raise the question if another types of solutions of Eqs. (355) $-$ (358) exist. If yes: Are there also such solutions whose curl do not vanish ?

3.4.4 The meaning of the vacuum solutions

The solutions presented in the previous sections 3.4.1 and 3.4.2 represent possible level II vacuum potentials and fields. We recall that according to Eqs. (361) $-$ (368) the levell II vacuum potentials and fields sum up to zero [32] at every location so that level I potentials and fields do not appear. The solutions presented in the sections 3.4.1 and 3.4.2 represent an infinite number of different electromagnetic vacuum potentials and fields. Within the framework of electro- and magnetostatics there are no obvious (boundary) conditions which specify a concrete type. Therefore the solutions presented in the sections 3.4.1 and 3.4.2 mean that electromagnetic vacuum potentials and fields are possible or exist, even if their concrete form remains an open question. Concerning this issue the following should be noted:

- The consideration of electromagnetic vacuum states within the framework of electro- and magnetostatics represents a rough approach and electrodynamics is certainly more appropriate to address this issue.
- The actual vacuum states are not only determined by electromagnetic potentials and fields but also by other contributions such as gravitational potentials and fields, and their mutual interaction.
- Even if the vacuum constitutes the overwhelming part of the universe, matter like electrically charged particles is also present. Therefore it seems likely that the actual vacuum potentials and fields are influenced by the presence of matter.

[32] The feature that the level I potentials and fields emerge from a difference of two level II potentials or fields, see e.g. Eqs. (319), (321), (298), (299) and (365) $-$ (368), can also be described as a sum of two quantities, for example $\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \phi(\vec{r}) + (-g(\vec{r}))$.

3.4.5 Hypothetical vacuum charge and current densities

According to Eqs. (361) $-$ (368) the level II vacuum potentials and fields sum up to zero [33] at every location so that level I potentials and fields do not appear [34]. Possibly, the presence of level II vacuum potentials and fields implies the existence of level II or vacuum charge and current densities, similar to the level I potentials and fields β , $\vec{A}$, $\vec{E}$ and $\vec{B}$ which are generated by the (level I) charge density ρ and current densitiy $\vec{J}$. The Eqs. (361) $-$ (368) suggest for the hypothetical vacuum charge densities, ρ_ϕ and ρ_g , and hypothetical vacuum current densities, $\vec{J}_A$ and $\vec{J}_V$, the relations [35]

$$\rho_g(\vec{r}) = -\rho_\phi(\vec{r}) \tag{418}$$

$$\vec{J}_V(\vec{r}) = -\vec{J}_A(\vec{r}) \tag{419}$$

so that the total (level I) charge density $\rho(\vec{r})$ and current density $\vec{J}(\vec{r})$ vanishes:

$$\rho(\vec{r}) = \rho_\phi(\vec{r}) + \rho_g(\vec{r}) = 0 \tag{420}$$

$$\vec{J}(\vec{r}) = \vec{J}_A(\vec{r}) + \vec{J}_V(\vec{r}) = 0 \tag{421}$$

It appears presently not clear how to compute the hypothetical vacuum charge and current density, $\rho_\phi(\vec{r})$ and $\vec{J}_A(\vec{r})$, from the vacuum potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$. One possibility is to assume that the relation between the hypothetical vacuum charge and current density and the vacuum potentials is of the type given by Eqs. (315) and (297). However, the decoupled linear seconder-order differential equations (315) and (297) describe the behavior of level I quantities, whereas the vacuum potentials are level II quantities which are specified by the partly coupled non-linear first-order differential equations (355) $-$ (359). Thus the relationship between the vacuum potentials and the hypothetical vacuum charge and current density remains an open question.

Furthermore, the consideration of electromagnetic vacuum states within the framework of electro- and magnetostatics represents a rough approach and electrodynamics is certainly more appropriate to address this issue.

[33] The feature that the level I potentials and fields emerge from a difference of two level II potentials or fields, see e.g. (319), (321), (298), (299) and (365) $-$ (368), can also be described as a sum of two quantities, for example $\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \phi(\vec{r}) + (-g(\vec{r}))$.

[34] The presence of (electrically charged) matter, i.e. charge density $\rho \neq 0$ and/or current density $\vec{J} \neq 0$, breaks this symmetry and level I potentials and fields emerge.

[35] The feature that the level I potentials and fields emerge from a difference of two level II potentials or fields, see e.g. (319), (321), (298), (299) and (365) $-$ (368), can also be described as a sum of two quantities, for example $\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \phi(\vec{r}) + (-g(\vec{r}))$.

3.5 The equations for the level II potentials in the absence of level I magnetic fields

In the following we present level II potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ which solve Eqs. (331) − (334) and (336) in the absence of a level I magnetic field, i.e. for $\vec{B}(\vec{r}) = 0$.

3.5.1 Presentation of a solution for any charge density

coming soon ...

3.6 A potential inconsistency in the equations and a brief list of some items whose consideration might lead to its elimination

coming soon ...

3.7 Summary

coming soon ...

4 The energy density of the fields and potentials

Recently H. Eckardt and D. W. Lindstrom have published a paper about the solutions of the latest set of electrodynamic ECE equations in the absence of level I electric and magnetic fields [25]. That paper comprises also a discussion about the energy density of the vacuum potentials [25].
In textbook electromagnetism the volumetric energy density $u_E(\vec{r})$ of the electric field $\vec{E}(\vec{r})$ and the volumetric energy density $u_B(\vec{r})$ of the magnetic field $\vec{B}(\vec{r})$ are given by

$$u_E(\vec{r}) = \frac{\epsilon_0}{2} \left| \vec{E}(\vec{r}) \right|^2 \tag{422}$$

$$u_B(\vec{r}) = \frac{1}{2\,\mu_0} \left| \vec{B}(\vec{r}) \right|^2 \tag{423}$$

The total energy density $u(\vec{r})$ is given by the sum

$$u(\vec{r}) = u_E(\vec{r}) + u_B(\vec{r}) = \frac{\epsilon_0}{2} \left| \vec{E}(\vec{r}) \right|^2 + \frac{1}{2\,\mu_0} \left| \vec{B}(\vec{r}) \right|^2 \tag{424}$$

In textbook electromagnetism the energy density u is related to the continuity equation and the electromagnetic energy momentum tensor $T^{\alpha\beta}$ whereby $T^{00} = u$.
The level I potentials and fields emerge from a difference or sum [36] between two

[36] The feature that the level I potentials and fields emerge from a difference of two level II potentials or fields can also be described as a sum of two quantities, for example
$\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r}) = \phi(\vec{r}) + (-g(\vec{r}))$.

associated level II potentials or fields which both depend on $\vec{r}$, see e.g. Eqs. (93), (94), (107) and (108):

- scalar potential $\beta(\vec{r}) = \phi(\vec{r}) - g(\vec{r})$
- electric field $\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r})$
- vector potential $\vec{\Lambda}(\vec{r}) = \vec{A}(\vec{r}) - \vec{V}(\vec{r})$
- magnetic field $\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r})$

In the case of the level II vacuum potentials or fields the corresponding difference or sum vanishes so that no level I potentials and fields appear, see e.g. Eqs. (146) – (149).

The presence of two associated but separate level II fields or potentials suggests the possibility that each of these two fields possesses its own energy density. If this is true, then the energy density $\tilde{u}_E(\vec{r})$ of the electric field

$$\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r}) \tag{425}$$

is given by the sum of the energy densitiy $u_\phi(\vec{r})$ of the field $\vec{E}_\phi(\vec{r})$ and the energy density $u_g(\vec{r})$ of the field $\vec{E}_g(\vec{r})$, i.e.

$$\tilde{u}_E(\vec{r}) = u_\phi(\vec{r}) + u_g(\vec{r}) \neq u_E(\vec{r}) \tag{426}$$

Analogously, if this is true, the energy density $\tilde{u}_B(\vec{r})$ of the magnetic field

$$\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r}) \tag{427}$$

is given by the sum of the energy densitiy $u_A(\vec{r})$ of the field $\vec{B}_A(\vec{r})$ and the energy density $u_V(\vec{r})$ of the field $\vec{B}_V(\vec{r})$, i.e.

$$\tilde{u}_B(\vec{r}) = u_A(\vec{r}) + u_V(\vec{r}) \neq u_B(\vec{r}) \tag{428}$$

If we assume that the energy density of the level II fields depends in the same way on the fields as the level I fields, see Eqs. (422) and (423), then we obtain from Eqs. (422), (423) and (425) – (428)

$$\tilde{u}_E(\vec{r}) \; = \; \frac{\epsilon_0}{2}\left|\vec{E}_\phi(\vec{r})\right|^2 + \frac{\epsilon_0}{2}\left|\vec{E}_g(\vec{r})\right|^2 \tag{429}$$

$$\begin{aligned} u_E(\vec{r}) \; &= \; \frac{\epsilon_0}{2}\left|\vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r})\right|^2 \\ &= \; \frac{\epsilon_0}{2}\left|\vec{E}_\phi(\vec{r})\right|^2 + \frac{\epsilon_0}{2}\left|\vec{E}_g(\vec{r})\right|^2 - \; \epsilon_0\, \vec{E}_\phi(\vec{r}) \cdot \vec{E}_g(\vec{r}) \end{aligned} \tag{430}$$

$$\tilde{u}_E(\vec{r}) \neq u_E(\vec{r}) \quad \text{for} \quad \vec{E}_\phi(\vec{r}) \cdot \vec{E}_g(\vec{r}) \neq 0 \tag{431}$$

and

$$\tilde{u}_B(\vec{r}) \;=\; \frac{1}{2\,\mu_0}\left|\vec{B}_A(\vec{r})\right|^2 + \frac{1}{2\,\mu_0}\left|\vec{B}_V(\vec{r})\right|^2 \tag{432}$$

$$\begin{aligned} u_B(\vec{r}) \;&=\; \frac{1}{2\,\mu_0}\left|\vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r})\right|^2 \\ &=\; \frac{1}{2\,\mu_0}\left|\vec{B}_A(\vec{r})\right|^2 + \frac{1}{2\,\mu_0}\left|\vec{B}_V(\vec{r})\right|^2 - \frac{1}{\mu_0}\,\vec{B}_A(\vec{r})\cdot\vec{B}_V(\vec{r}) \end{aligned} \tag{433}$$

$$\tilde{u}_B(\vec{r}) \neq u_B(\vec{r}) \quad \text{for} \quad \vec{B}_A(\vec{r})\cdot\vec{B}_V(\vec{r}) \neq 0 \tag{434}$$

By inserting Eqs. (95) and (96) into Eqs. (429) and (430) and Eqs. (109) and (110) into Eqs. (432) and (433) we obtain the energy densities in terms of the potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$, namely

$$\tilde{u}_E(\vec{r}) \;=\; \frac{\epsilon_0}{2}\left|\vec{\nabla}\phi\right|^2 + \frac{\epsilon_0}{2}\left|\vec{\nabla}g\right|^2 \tag{435}$$

$$\begin{aligned} u_E(\vec{r}) \;&=\; \frac{\epsilon_0}{2}\left|\vec{\nabla}\,(\phi - g)\right|^2 \\ &=\; \frac{\epsilon_0}{2}\left|\vec{\nabla}\phi\right|^2 + \frac{\epsilon_0}{2}\left|\vec{\nabla}g\right|^2 - \epsilon_0\left(\vec{\nabla}\phi\right)\cdot\left(\vec{\nabla}g\right) \end{aligned} \tag{436}$$

$$\tilde{u}_E(\vec{r}) \neq u_E(\vec{r}) \quad \text{for} \quad \left(\vec{\nabla}\phi\right)\cdot\left(\vec{\nabla}g\right) \neq 0 \tag{437}$$

and

$$\tilde{u}_B(\vec{r}) \;=\; \frac{1}{2\,\mu_0}\left|\vec{\nabla}\times\vec{A}\right|^2 + \frac{1}{2\,\mu_0}\left|\vec{\nabla}\times\vec{V}\right|^2 \tag{438}$$

$$\begin{aligned} u_B(\vec{r}) \;&=\; \frac{1}{2\,\mu_0}\left|\vec{\nabla}\times\left(\vec{A} - \vec{V}\right)\right|^2 \\ &=\; \frac{1}{2\,\mu_0}\left|\vec{\nabla}\times\vec{A}\right|^2 + \frac{1}{2\,\mu_0}\left|\vec{\nabla}\times\vec{V}\right|^2 \\ &\quad - \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{A}\right)\cdot\left(\vec{\nabla}\times\vec{V}\right) \end{aligned} \tag{439}$$

$$\tilde{u}_B(\vec{r}) \neq u_B(\vec{r}) \quad \text{for} \quad \left(\vec{\nabla}\times\vec{A}\right)\cdot\left(\vec{\nabla}\times\vec{V}\right) \neq 0 \tag{440}$$

We note that the absence of a level I electric field, i.e.

$$\vec{E}(\vec{r}) = \vec{E}_\phi(\vec{r}) - \vec{E}_g(\vec{r}) = -\vec{\nabla}(\phi - g) = 0\,, \tag{441}$$

implies

$$\vec{E}_\phi(\vec{r}) = \vec{E}_g(\vec{r}) \tag{442}$$

$$\vec{\nabla}\phi = \vec{\nabla}g \tag{443}$$

$$u_E(\vec{r}) = 0 \tag{444}$$

$$\tilde{u}_E(\vec{r}) \;=\; \epsilon_0 \left|\vec{E}_\phi(\vec{r})\right|^2 \;=\; \epsilon_0 \left|\vec{\nabla}\phi\right|^2 \neq\; 0 \tag{445}$$

Analogously, the absence of a level I magnetic field, i.e.

$$\vec{B}(\vec{r}) = \vec{B}_A(\vec{r}) - \vec{B}_V(\vec{r}) = \vec{\nabla} \times \left(\vec{A} - \vec{V}\right) \;=\; 0\,, \tag{446}$$

implies

$$\vec{B}_A(\vec{r}) = \vec{B}_V(\vec{r}) \tag{447}$$

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{V} \tag{448}$$

$$u_B(\vec{r}) = 0 \tag{449}$$

$$\tilde{u}_B(\vec{r}) \;=\; \frac{1}{\mu_0}\left|\,\vec{B}_A(\vec{r})\,\right|^2 \;=\; \frac{1}{\mu_0}\left|\vec{\nabla} \times \vec{A}\right|^2 \neq\; 0 \tag{450}$$

Does $\tilde{u}(\vec{r})$ or $u(\vec{r})$ represent the appropriate energy density ? It appears reasonable to assume that $\tilde{u}(\vec{r})$ is the appropriate energy density. If this is true, then Eqs. (445) and (450) represent the energy density of level II vacuum potentials in the absence of level I fields.

We suggest to study in a separate theoretical work the question if $\tilde{u}(\vec{r})$ or $u(\vec{r})$ is the appropriate energy density. Concerning this issue let's consider the following gedanken experiment. Let's imagine a bifilar wire which consists, for example, of a cylindrical hollow conductor and a cylindrical solid interior conductor. Assuming that a DC current I_1 flows through one conductor and another DC current I_2 flows in an opposite direction through the other conductor. The current I_1 creates an external magnetic field $\vec{B}_1$, the current I_2 an external magnetic field $\vec{B}_2$, and the total external magnetic field $\vec{B}$ is given by

$$\vec{B}(I_1\,, I_2\,, \vec{r}) \;=\; \vec{B}_1(I_1\,, \vec{r}) \;+\; \vec{B}_2(I_2\,, \vec{r}) \tag{451}$$

This raises the question if the energy density is given by the total magnetic field $\vec{B}$, i.e.

$$\begin{aligned} u_B(I_1\,, I_2\,, \vec{r}\,) \;&=\; \frac{1}{2\,\mu_0}\,\left|\vec{B}\,(I_1\,, I_2\,, \vec{r}\,)\right|^2 \\ &=\; \frac{1}{2\,\mu_0}\,\left|\vec{B}_1\,(I_1\,, \vec{r}\,) + \vec{B}_2\,(I_2\,, \vec{r}\,)\right|^2 \\ &=\; \frac{1}{2\,\mu_0}\,\left|\vec{B}_1\,(I_1\,, \vec{r}\,)\right|^2 + \;\frac{1}{2\,\mu_0}\,\left|\vec{B}_2\,(I_2\,, \vec{r}\,)\right|^2 \\ &\quad + \frac{1}{\mu_0}\;\vec{B}_1\,(I_1\,, \vec{r}\,)\cdot\vec{B}_2\,(I_2\,, \vec{r}\,) \end{aligned} \tag{452}$$

or by the sum of the energy densities of the separate fields $\vec{B}_1$ and $\vec{B}_2$, i.e.

$$\begin{aligned} u_{12}\,(I_1\,, I_2\,, \vec{r}\,) \;&=\; u_1\Big(\vec{B}_1\,(I_1\,, \vec{r}\,)\Big) \;+\; u_2\Big(\vec{B}_2\,(I_2\,, \vec{r}\,)\Big) \\ &=\; \frac{1}{2\,\mu_0}\,\left|\vec{B}_1\,(I_1\,, \vec{r}\,)\right|^2 + \;\frac{1}{2\,\mu_0}\,\left|\vec{B}_2\,(I_2\,, \vec{r}\,)\right|^2 \end{aligned} \tag{453}$$

and thus

$$u_B(I_1\,, I_2\,, \vec{r}\,) \neq u_{12}\,(I_1\,, I_2\,, \vec{r}\,) \quad \text{for} \quad \vec{B}_1\,(I_1\,, \vec{r}\,)\cdot\vec{B}_2\,(I_2\,, \vec{r}\,) \neq 0 \tag{454}$$

The question if $u_B(I_1\,, I_2\,, \vec{r}\,)$ or $u_{12}\,(I_1\,, I_2\,, \vec{r}\,)$ represents the appropriate energy density remains open for further discussions and studies. This question is especially interesting for $\vec{B} = 0$ which implies $u_B = 0$ and $u_{12} \neq 0$. A vanishing total field, i.e. $\vec{B} = 0$, can be achieved for special values of the opposite flowing currents I_1 and I_2 , at least for certain spatial positions.

We note that $u_B = u_{12}$ if $\vec{B}_1 \cdot \vec{B}_2 = 0$, i.e. if $\vec{B}_1$ is perpendicular to $\vec{B}_2$. This, however, is not the case in our gedanken experiment because the considered bifilar wire with opposite flowing currents implies that the external fields $\vec{B}_1\,(I_1\,, \vec{r}\,)$ and $\vec{B}_2\,(I_2\,, \vec{r}\,)$ are antiparallel to each other.

5 A brief consideration of the electrodynamic ECE equations

Even if this paper is dedicated to the electro- and magnetostatic ECE equations we also present here the time-dependent equations of ECE electrodynamics, especially because we raise the question if the electrodynamic ECE equations can be transformed and studied in a similar way like the electro- and magnetostatic ECE equations.

The latest set of equations of ECE electrodynamics in vector notation, see e.g. Ref. [6], is given by [37]

[37] Eqs. (455) − (464) refer to the assumption that the so-called polarization index can be omitted, i.e. one polarization only, see e.g. Ref. [6].

Antisymmetry constraints:

$$\vec{\nabla}\phi \,-\, \phi\,\vec{\omega} \,-\, \omega_0\,\vec{A} \,-\, \frac{\partial}{\partial\,t}\,\vec{A} \,=\, 0 \tag{455}$$

$$\frac{\partial A_3}{\partial\,y} + \frac{\partial A_2}{\partial\,z} + \omega_2 A_3 + \omega_3 A_2 = 0 \tag{456}$$

$$\frac{\partial A_3}{\partial\,x} + \frac{\partial A_1}{\partial\,z} + \omega_1 A_3 + \omega_3 A_1 = 0 \tag{457}$$

$$\frac{\partial A_2}{\partial\,x} + \frac{\partial A_1}{\partial\,y} + \omega_1 A_2 + \omega_2 A_1 = 0 \tag{458}$$

whereby $\vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ and $\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$

Field equations in terms of potentials:

Gauss law:

$$\vec{\nabla} \cdot \left(\vec{\omega} \times \vec{A} \right) \,=\, 0 \tag{459}$$

Faraday induction law:

$$\vec{\nabla} \times \left(\phi\,\vec{\omega} \,-\, \omega_0\,\vec{A} \right) - \frac{\partial}{\partial\,t} \left(\vec{\omega} \times \vec{A} \right) = 0 \tag{460}$$

Coulomb law:

$$\Delta\phi \,-\, \vec{\nabla} \cdot \left(\phi\,\vec{\omega} \,-\, \omega_0\,\vec{A} \right) + \vec{\nabla} \cdot \left(\frac{\partial}{\partial\,t}\,\vec{A} \right) = -\frac{\rho}{\epsilon_0} \tag{461}$$

Ampere-Maxwell law:

$$\begin{aligned} &\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} - \vec{\omega} \times \vec{A} \right) \\ &+ \frac{1}{c^2} \left[\frac{\partial^2}{\partial\,t^2}\,\vec{A} \,+\, \vec{\nabla}\,\frac{\partial\,\phi}{\partial\,t} \,-\, \frac{\partial}{\partial\,t} \left(\phi\,\vec{\omega} \,-\, \omega_0\,\vec{A} \right) \right] \,=\, \mu_0\,\vec{J} \end{aligned} \tag{462}$$

Field-potential relations:

$$\vec{B} \,=\, \vec{\nabla} \times \vec{A} \,-\, \vec{\omega} \times \vec{A} \tag{463}$$

$$\vec{E} \,=\, -\,\vec{\nabla}\phi \,+\, \phi\,\vec{\omega} - \omega_0\,\vec{A} - \frac{\partial}{\partial\,t}\,\vec{A} \tag{464}$$

The electric field $\vec{E}$ can also be represented in another way. By solving Eq. (455) for

$\vec{\nabla}\phi$ and inserting it into Eq. (464) we get

$$\vec{E} = -\,2\left(\omega_0\,\vec{A} \,-\, \frac{\partial}{\partial\,t}\,\vec{A}\right) \tag{465}$$

Another expression for the electric field $\vec{E}$ can be obtained by solving Eq. (455) for $\frac{\partial}{\partial\,t}\vec{A}$ and inserting it into Eq. (464). This leads to

$$\vec{E} = -2\left(\vec{\nabla}\phi \,-\, \phi\,\vec{\omega}\right) \tag{466}$$

According to Ref. [13] the electrodynamic ECE equations (455) $-$ (464) merge into the Maxwell or Maxwell-Heaviside equations of textbook electrodynamics if [38]

$$\phi\,\vec{\omega} = \omega_0\,\vec{A} \tag{467}$$

which implies $\vec{\omega}\times\vec{A} = 0$ because in this case $\vec{\omega}$ and $\vec{A}$ are (anti)parallel to each other. As shown in Ref. [14] there are special types of scalar potentials ϕ which are in accordance with Eq. (467), namely [39]

$$\phi(\vec{r},\,t) = \phi_t(t)\,\phi_r(\vec{r}) \quad\Rightarrow\quad \phi(\vec{r},\,t) = \beta(\vec{r},\,t) \tag{468}$$

whereby $\beta(\vec{r},\,t)$ is the scalar potential of textbook electrodynamics. Thus, if the scalar potential $\phi(\vec{r},\,t)$ is given by a product of a time-dependent function $\phi_t(t)$ and a position-dependent function $\phi_r(\vec{r})$, then the electrodynamic ECE equations (455) $-$ (464) merge into the Maxwell or Maxwell-Heaviside equations of textbook electrodynamics [14]. This means that novel effects, i.e. such which are beyond textbook electrodynamics, can only be expected if the potential $\phi(\vec{r},\,t)$ cannot be represented by a product of a time-dependent and a position-dependent function. This statement is based on the assumption that $\phi(\vec{r},\,t)$ represents the physically relevant potential which results in observable effects. However, the considerations of the electro- and magnetostatic ECE equations lead to the question if the electro- and magnetostatic level II potentials $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ are the physically relevant potentials which result in observable effects. Therefore we raise the same question for the electrodynamic case: Are the time-dependent potentials $\phi(\vec{r},\,t)$ and $\vec{A}(\vec{r},\,t)$ the relevant potentials which result in observable effects ? We suggest to study in a separate work the question if the electrodynamic ECE equations can be transformed and considered in a similar way like the electro- and magnetostatic ECE equations. A hypothetical scenario, suggested by the results of the studies of the static case, could be the following:

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[38] For $\vec{\omega} = 0$ and $\omega_0 = 0$, and by omitting the antisymmetry constraints (455) $-$ (458), the electrodynamic ECE equations (455) $-$ (464) merge likewise into the Maxwell or Maxwell-Heaviside equations of textbook electrodynamics.

[39] If the scalar potential $\phi(\vec{r},\,t)$ is of the type Eq. (468), then also the vector potential $\vec{A}(\vec{r},\,t)$ is constitued by a product of a time-dependent and a position-dependent function [14].

Possibly there is a transformation of the electrodynamic ECE equations which reveals the existence of potentials $\widetilde{\beta}(\vec{r},t)$ and $\widetilde{\vec{\Lambda}}(\vec{r},t)$ that emerge from a difference of two potentials which depend both on time and spatial location, i.e.

$$\widetilde{\beta}(\vec{r},t) = \phi(\vec{r},t) - g(\vec{r},t) \tag{469}$$

$$\widetilde{\vec{\Lambda}}(\vec{r},t) = \vec{A}(\vec{r},t) - \vec{V}(\vec{r},t) \tag{470}$$

If this turns out to be true, then possibly $\widetilde{\beta}(\vec{r},t)$ and $\widetilde{\vec{\Lambda}}(\vec{r},t)$ are the physically relevant potentials and novel effects, i.e. such which are beyond textbook electrodynamics, appear under special conditions which imply

$$\widetilde{\beta}(\vec{r},t) \;\neq\; \beta(\vec{r},t) \tag{471}$$

$$\widetilde{\vec{\Lambda}}(\vec{r},t) \;\neq\; \vec{\Lambda}(\vec{r},t) \tag{472}$$

$$\vec{E}(\vec{r},t) \;\neq\; \vec{E}_T(\vec{r},t) \tag{473}$$

$$\vec{B}(\vec{r},t) \;\neq\; \vec{B}_T(\vec{r},t) \tag{474}$$

whereby $\beta(\vec{r},t)$ and $\vec{\Lambda}(\vec{r},t)$ are the potentials and $\vec{E}_T(\vec{r},t)$ and $\vec{B}_T(\vec{r},t)$ the fields of textbook electrodynamics. However, as already mentioned, further studies are necessary to clarify if this is really true or not.

Concerning the vector potential and magnetic field it is readily obvious that a relation of the type of Eq. (470) exists. Analogous to the electro- and magnetostatic equations, see sections 2.2 and 3.2, we infer from Eq. (459) that $\vec{\omega} \times \vec{A}$ can be written as

$$\vec{\omega} \times \vec{A} = \vec{\nabla} \times \vec{V} \tag{475}$$

whereby $\vec{V} = \vec{V}(\vec{r},t)$ is another vector potential which depends in some way on the vector potential $\vec{A} = \vec{A}(\vec{r},t)$. Thus from Eqs. (475) and (463) we obtain

$$\vec{B} \;=\; \vec{\nabla} \times \vec{A} \;-\; \vec{\nabla} \times \vec{V} = \vec{\nabla} \times \left(\vec{A} - \vec{V}\right) \;=\; \vec{\nabla} \times \widetilde{\vec{\Lambda}} \tag{476}$$

whereby

$$\widetilde{\vec{\Lambda}}(\vec{r},t) \;=\; \vec{A}(\vec{r},t) - \vec{V}(\vec{r},t) \tag{477}$$

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